Integrating Efficiency Concepts in Technology Approximation: A Weighted DEA Approach

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Abstract: A method is developed to integrate the concepts of technical, scale, and allocative inefficiencies (TI, SI, and AI) into a frontier approximation under Data Envelopment Analysis (DEA). The proposed weighted DEA (WDEA) approach employs a weighted average of the three benchmarking frontiers associated with the three efficiency concepts, extending the conventional frontier approximation under variable returns to scale (VRS) toward scale- and allocatively-efficient decisions. A weight selection rule is devised based on a finite-sample bias correction technique and sample correlations between the underestimation of TI and the overestimation of SI and AI under conventional measures. The WDEA approach is consistent and more efficient than the VRS model under the maintained properties of data generating process. An application to U.S. dairy production data finds that on average TI is 13 to 21 percentage-point higher under WDEA than under VRS, depending on the geographic region.

Keywords: Data envelopment analysis, Technical Efficiency, Scale Efficiency, Allocative Efficiency, Production Economics

JEL *Codes*: D22, Q12, C44

1 Introduction

The concept of optimality in decision-making is defined by a relevant "frontier" of decision possibilities. The production economics theory offers three major concepts of optimality in the decision space. Technical inefficiency (TI) assesses the extent of feasible output expansions for given inputs (or input reductions for given outputs) relative to the *frontier* of technically-feasible input-output decisions. Scale and allocative inefficiencies (SI and AI) represent the extents of forgone opportunities due to suboptimal scales of operations and misallocation of resources, relative to the *frontier* of linear-homogeneous production process (i.e. constant returns to scale; CRS) and the *frontier* of revenue maximization (or cost minimization) respectively. A large body

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of empirical research has analyzed technological frontier and TI in various contexts, whereas studies investigating SI or AI have been scarce.

However, the interconnections among the concepts of TI, SI, and AI suggest an opportunity to improve technological frontier estimation. Conceptually, TI is a gap between an input-output decision and a technological frontier, and AI and SI are the gaps between the technological frontier and its outer "frontiers" of different benchmarking concepts. The pivotal role played by the technological frontier implies that the most efficient estimation strategy entails a joint specification of the frontier and these inefficiency concepts. Such estimation methods have been developed in the parametric frontier literature, for instance, through incorporating AI into the optimal factor demands from cost minimization (e.g., Yotopoulos and Lau, 1973; Schmidt and Lovell, 1979, 1980; Kumbhakar, 1989, 1997; Kumbhakar and Wang, 2006; Kumbhakar and Tsionas, 2011). In the semi-parametric frontier literature that includes models like Data Envelopment Analysis (DEA), on the other hand, there is no coherent estimation technique that integrates these efficiency concepts. This article takes a step toward filling this gap of knowledge.

Inefficient DEA estimation manifests itself in the form of a limited ability to discriminate individual TI measurements. DEA analyses on small samples tend to find an unexpectedly large number of observations being fully technically-efficient, a frequent concern in this literature (e.g., Dyson et al., 2001; Podinovski and Thanassoulis, 2007). Some researchers have tackled this issue via shadow price restrictions, or direct value judgements passed through perceived importance of inputs and outputs (e.g., Allen et al., 1997; Thanassoulis, Portela, and Allen, 2004) or so-called assurance regions (e.g., Dyson and Thanassoulis, 1988; Thompson et al., 1990; Sarrico and Dyson, 2004; Podinovski, 2004a; Tracy and Chen, 2004; Khalili et al., 2010). Others have approached from a perspective of technology parameter restrictions that incorporates expert knowledge on the production processes. Such examples include weak disposability of inputs or undesirable outputs (Chung, Fre, and Grosskopf, 1997; Scheel, 2001; Seiford and Zhu, 2002; Kuosmanen, 2005; Podinovski and Kuosmanen, 2011), non-discretionary factors (Ruggiero, 1998), unobserved decisions (Thanassoulis and Allen, 1998; Allen and Thanassoulis, 2004), selective linear homogeneity (Podinovski, 2004b; Podinovski and Thanassoulis, 2007), and prescribed producer trade-offs (Podinovski, 2004c). Following the latter line of research, this article proposes to refine a variable returns to scale (VRS) frontier estimation through accommodating additional degrees of linear homogeneity and technical substitution, based on the sample distributions of conventional TI, SI, and AI measures. The method is a variant of the DEA frontier bounds of Chambers, Chung, and Färe (1998) and closely related to the allocative inefficiency bounds of Kuosmanen and Post (2001).

The proposed weighted DEA (WDEA) approach below estimates a technological frontier as a weighted average of the three benchmarking-frontier concepts of TI, SI, and AI and extends the conventional VRS frontier toward scale- and allocatively-efficient decisions. A weight selection rule is devised based on a finite-sample bias correction technique and sample correlations between the underestimation of TI and the overestimation of SI and AI under conventional measures. The resulting WDEA frontier is consistent and more efficient than the VRS frontier under the maintained properties of data generating process and enhances the discriminatory power of DEA.

In the following, section 2 presents the WDEA approach, and section 3 demonstrates the method in an application to U.S. dairy production data, followed by conclusions in section 4.

2 Methods

2.1 Preliminaries

Preliminary concepts are defined as follows. Technology T is a set of feasible input-output bundles, or $T = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}_+^L \times \mathbb{R}_+^M : \ \boldsymbol{x} \text{ can produce } \boldsymbol{y}\}$ where

A.1 T is closed.
A.2 T satisfies free-disposability: $(\boldsymbol{x}, \boldsymbol{y}) \in T$ and $(-\boldsymbol{x}, \boldsymbol{y}) \geq (-\boldsymbol{x}', \boldsymbol{y}') \ \Rightarrow \ (\boldsymbol{x}', \boldsymbol{y}') \in T$.
A.3 T is convex: $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}') \in T \ \Rightarrow \forall\lambda \in [0, 1], \ \forall(\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{x}', \lambda\boldsymbol{y} + (1-\lambda)\boldsymbol{y}') \in T$.

The boundary of technology T is referred to as *technological frontier*. T can be completely characterized by the directional distance function of Chambers, Chung, and Färe (1998)[1] in the sense that $(\boldsymbol{x}, \boldsymbol{y}) \in T \Leftrightarrow TI_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) \geq 0$ where

$$TI_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) = \max\{b \in \mathbb{R} : (\boldsymbol{x} - b\boldsymbol{g}_x, \boldsymbol{y} + b\boldsymbol{g}_y) \in T\}. \tag{1}$$

Function $TI_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y)$ measures the distance between point $(\boldsymbol{x}, \boldsymbol{y})$ and the frontier of technology T in direction $(\boldsymbol{g}_x, \boldsymbol{g}_y)$ that represents technical inefficiency (TI). As a special case, setting direction $(\boldsymbol{g}_x, \boldsymbol{g}_y) = (\boldsymbol{x}_0, \boldsymbol{0})$ yields the input-oriented, radial TI measurement, which is equivalent to Shephard's input distance function $\theta_V(\boldsymbol{x}_0, \boldsymbol{y}_0) = \max\{\theta : \boldsymbol{x}_0/\theta \in V(\boldsymbol{y}_0)\} \geq 1$ for input set $V(\boldsymbol{y})$ of technology T in that $\theta_V(.) = 1/(1 - TI_T(.))$. Similarly, setting direction $(\boldsymbol{g}_x, \boldsymbol{g}_y) = (\boldsymbol{0}, \boldsymbol{y}_0)$ leads to the output-oriented, radial TI measurement, or Farrell's output efficiency $\phi_Y(\boldsymbol{x}_0, \boldsymbol{y}_0) = \max\{\phi : \phi\boldsymbol{y}_0 \in Y(\boldsymbol{x}_0)\} \geq 1$ for output set $Y(\boldsymbol{x})$ of technology T with $\phi_Y(.) = 1 + TI_T(.)$.

[1]The standard notation for the directional distance function is $D(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y)$. It is the technology-counterpart to the shortage function of Luenberger (1994).

The duality between the profit function and the directional distance function states;

$$\begin{aligned}\pi_T(\boldsymbol{w},\boldsymbol{p}) &= \max_{\boldsymbol{x},\boldsymbol{y}}\{\boldsymbol{p}\boldsymbol{y}-\boldsymbol{w}\boldsymbol{x}:\ (\boldsymbol{x},\boldsymbol{y})\in T\}\\ &= \max_{\boldsymbol{x},\boldsymbol{y}}\{\boldsymbol{p}\boldsymbol{y}-\boldsymbol{w}\boldsymbol{x}+TI_T(\boldsymbol{x},\boldsymbol{y};\boldsymbol{g}_x,\boldsymbol{g}_y)(\boldsymbol{p}\boldsymbol{g}_y+\boldsymbol{w}\boldsymbol{g}_x)\}\end{aligned} \tag{2}$$

and

$$TI_T(\boldsymbol{x},\boldsymbol{y};\boldsymbol{g}_x,\boldsymbol{g}_y) = \min_{\boldsymbol{w}',\boldsymbol{p}'}\left\{\frac{\pi_T(\boldsymbol{p}',\boldsymbol{w}')-(\boldsymbol{p}'\boldsymbol{y}-\boldsymbol{w}'\boldsymbol{x})}{\boldsymbol{p}'\boldsymbol{g}_y+\boldsymbol{w}'\boldsymbol{g}_x}\right\} \tag{3}$$

where the profit function $\pi_T(\boldsymbol{w},\boldsymbol{p})$ attains the highest production value in technology T for given input-output prices $(\boldsymbol{w},\boldsymbol{p})\in\mathbb{R}_+^{L+M}$. The second line in (2) follows from the definition of directional distance function $(\boldsymbol{x}-TI_T(.)\boldsymbol{g}_x,\ \boldsymbol{y}+TI_T(.)\boldsymbol{g}_y)\in T$. Equation (3) is a dual expression of (1), showing that TI can be cast as an evaluation of the decision $(\boldsymbol{x},\boldsymbol{y})$ against its maximum potential $\pi_T(\boldsymbol{p}',\boldsymbol{w}')$ under its shadow prices $(p',\boldsymbol{w}')$ (i.e., a supporting hyperplane that shows a given decision in the most favorable light).

Consider two additional concepts of frontiers. The assumption of constant returns to scale (CRS) considers a hypothetical technology that envelops T under the linear homogeneity of input-output relationships, or

$$T_{CRS} = \cup_{\lambda\in\mathbb{R}_+}\lambda\, T. \tag{4}$$

The associated profit and distance functions can be stated as;

$$\pi_{CRS}(\boldsymbol{w},\boldsymbol{p}) = \max_{\boldsymbol{x},\boldsymbol{y}}\{\boldsymbol{p}\boldsymbol{y}-\boldsymbol{w}\boldsymbol{x}:\ (\boldsymbol{x},\boldsymbol{y})\in T_{CRS}\}, \tag{5}$$

$$TI_{CRS}(\boldsymbol{x},\boldsymbol{y};\boldsymbol{g}_x,\boldsymbol{g}_y) = \min_{\boldsymbol{w}',\boldsymbol{p}'}\left\{\frac{\pi_{CRS}(\boldsymbol{w}',\boldsymbol{p}')-(\boldsymbol{p}'\boldsymbol{y}-\boldsymbol{w}'\boldsymbol{x})}{\boldsymbol{p}'\boldsymbol{g}_y+\boldsymbol{w}'\boldsymbol{g}_x}\right\} \tag{6}$$

where in theory $\pi_{CRS}(\boldsymbol{w},\boldsymbol{p})$ is either 0 (e.g., free entry and exit) or ∞ (e.g., limited entry and an arbitrarily large demand for the output), and $TI_{CRS}(.)$ is the pseudo-TI measurement against T_{CRS}.

Let profit-frontier (PF) technology $T_{PF}(\boldsymbol{w},\boldsymbol{p})$ be a hypothetical technology that further envelopes T_{CRS} under linear technical substitutability and takes the form of a half-space defined by given market prices $(\boldsymbol{w},\boldsymbol{p})$ and bounded by the profit function, or

$$T_{PF}(\boldsymbol{w},\boldsymbol{p}) = \{(\boldsymbol{x},\boldsymbol{y}):\ \boldsymbol{p}\boldsymbol{y}-\boldsymbol{w}\boldsymbol{x}\le\pi_{CRS}(\boldsymbol{w},\boldsymbol{p})\}. \tag{7}$$

Under output- or input-oriented efficiency measurement (i.e., $(\boldsymbol{g}_x, \boldsymbol{g}_y) = (\boldsymbol{0}, \boldsymbol{g}_y)$ or $(\boldsymbol{g}_x, \boldsymbol{g}_y) = (\boldsymbol{g}_x, \boldsymbol{0})$), $T_{PF}(\boldsymbol{w}, \boldsymbol{p})$ reduces to the half-space bounded by the revenue- or cost-frontier respectively.

Now let profit inefficiency (PI) be

$$PI(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{w}, \boldsymbol{p}, \boldsymbol{g}_x, \boldsymbol{g}_y) = \frac{\pi_{CRS}(\boldsymbol{w}, \boldsymbol{p}) - (\boldsymbol{py} - \boldsymbol{wx})}{\boldsymbol{pg}_y + \boldsymbol{wg}_x}, \tag{8}$$

which evaluates the decision $(\boldsymbol{x}, \boldsymbol{y})$ against its maximum potential $\pi_{CRS}(\boldsymbol{p}, \boldsymbol{w})$ under prices $(\boldsymbol{w}, \boldsymbol{p})$ and normalization $\boldsymbol{pg}_y + \boldsymbol{wg}_x$. The PI in (8) can be additively decomposed into three types of inefficiencies associated with technical conversion of inputs to outputs (TI), operational scales (Scale Inefficiency: SI), and resource allocation (Allocative Inefficiency: AI)[2];

$$\begin{aligned} PI &= TI + SI + AI \\ &= TI_T + (TI_{CRS} - TI_T) + (PI - TI_{CRS}). \end{aligned} \tag{9}$$

where $TI \equiv TI_T$, $SI \equiv TI_{CRS} - TI_T$, $AI \equiv PI - TI_{CRS}$. Formally, the SI and AI are stated as;

$$SI(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) = TI_{CRS}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) - TI_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y), \tag{10}$$

$$AI(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{w}, \boldsymbol{p}, \boldsymbol{g}_x, \boldsymbol{g}_y) = PI(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{w}, \boldsymbol{p}, \boldsymbol{g}_x, \boldsymbol{g}_y) - TI_{CRS}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y). \tag{11}$$

$SI(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) \geq 0$ by equation (1) and $T \subset T_{CRS}$, and $AI(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{w}, \boldsymbol{p}, \boldsymbol{g}_x, \boldsymbol{g}_y) \geq 0$ by equation (6). Any decision on the frontier of T_{CRS} is scale-efficient, and any decision on the frontier of T_{PF} is allocatively-efficient.

2.2 Weighted DEA (WDEA) Approach

The weighted DEA (WDEA) approach below operationalizes the decomposition in equation (9). The empirical case is denoted as input-output data $\{(\boldsymbol{x}_i, \boldsymbol{y}_i)\}_{i \in \mathbb{I}}$ with observation index set $\mathbb{I} = \{1...N\}$.

The DEA approximations under VRS and CRS are respectively the free-disposal convex hull of data points (i.e., all convex combinations of data points and the points implied by free-disposability) and the free-disposal conical hull of data points (i.e., every point in $\widehat{T}_{VRS}$ and any

[2] In his seminal study, Nerlove (1965) introduced profit efficiency defined as $(\boldsymbol{py} - \boldsymbol{wx})/\pi(\boldsymbol{w}, \boldsymbol{p})$ and multiplicatively decomposed it into technical and allocative efficiencies.

scaler multiple of it), or

$$\widehat{T}_{VRS} = \{(\boldsymbol{x}', \boldsymbol{y}') : \sum_{j \in \mathbb{I}} \lambda_j \boldsymbol{y}_j \geq \boldsymbol{y}', \ , \sum_{j \in \mathbb{I}} \lambda_j \boldsymbol{x}_j \leq \boldsymbol{x}', \ \sum_{j \in \mathbb{I}} \lambda_j = 1, \ \boldsymbol{\lambda} \in \mathbb{R}^N_+\}, \tag{12}$$

$$\widehat{T}_{CRS} = \{(\boldsymbol{x}', \boldsymbol{y}') : \sum_{j \in \mathbb{I}} \lambda_j \boldsymbol{y}_j \geq \boldsymbol{y}', \ , \sum_{j \in \mathbb{I}} \lambda_j \boldsymbol{x}_j \leq \boldsymbol{x}', \ \boldsymbol{\lambda} \in \mathbb{R}^N_+\}. \tag{13}$$

$\widehat{T}_{VRS}$ corresponds to the smallest producible set satisfying assumptions A.1-A.3, while $\widehat{T}_{CRS}$ envelops $\widehat{T}_{VRS}$ under linear homogeneity.

The estimation of TI_T and TI_{CRS} by the directional distance function yields $\widehat{TI}_{VRS}(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{g}_x, \boldsymbol{g}_y) = \max\{b : (\boldsymbol{x}_0 - b\boldsymbol{g}_x, \boldsymbol{y}_0 + b\boldsymbol{g}_y) \in \widehat{T}_{VRS}\}$ and $\widehat{TI}_{CRS}(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{g}_x, \boldsymbol{g}_y) = \max\{b : (\boldsymbol{x}_0 - b\boldsymbol{g}_x, \boldsymbol{y}_0 + b\boldsymbol{g}_y) \in \widehat{T}_{CRS}\}$ respectively. The dual problem for $\widehat{TI}_{VRS}(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{g}_x, \boldsymbol{g}_y)$, corresponding to the dual representation in (3), is

$$\begin{aligned} \min\{\rho \in \mathbb{R} : \ & \forall j \in \mathbb{I}, \ \boldsymbol{p}'\boldsymbol{y}_j - \boldsymbol{w}'\boldsymbol{x}_j \leq \boldsymbol{p}'\boldsymbol{y}_0 - \boldsymbol{w}'\boldsymbol{x}_0 + \rho, \\ & \boldsymbol{p}'\boldsymbol{g}_y + \boldsymbol{w}'\boldsymbol{g}_x = 1, \ \boldsymbol{p}' \in \mathbb{R}^M_+, \ \boldsymbol{w}' \in \mathbb{R}^L_+\}, \end{aligned} \tag{14}$$

which minimizes TI-parameter ρ subject to the optimality of shadow value $\boldsymbol{p}'\boldsymbol{y}_0 - \boldsymbol{w}'\boldsymbol{x}_0 + \rho$ for decision $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ relative to other data points $\boldsymbol{p}'\boldsymbol{y}_j - \boldsymbol{w}'\boldsymbol{x}_j$, $\forall j \in \mathbb{I}$ under shadow-price normalization $\boldsymbol{p}'\boldsymbol{g}_y + \boldsymbol{w}'\boldsymbol{g}_x = 1$. Imposing additional constraint $\boldsymbol{p}'\boldsymbol{y}_0 - \boldsymbol{w}'\boldsymbol{x}_0 + \rho \leq 0$ in problem (14), as implied by condition $\pi_{CRS}(\boldsymbol{p}', \boldsymbol{w}') = 0$ (i.e., $\pi_{CRS}(\boldsymbol{p}', \boldsymbol{w}') \neq \infty$), yields the dual estimation for $\widehat{TI}_{CRS}(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{g}_x, \boldsymbol{g}_y)$ corresponding to (6).

For practical purposes, the computation of PI in (8) may be based on an estimate of $\pi_{CRS}(\boldsymbol{p}, \boldsymbol{w})$ at prices $(\boldsymbol{p}, \boldsymbol{w})$, which is neither infinite nor zero. For instance, the maximum profit level given data can be predicted using TI_{CRS};

$$\widehat{\pi}^*_{CRS}(\boldsymbol{p}, \boldsymbol{w}) = \max_{j \in \mathbb{I}} \{\boldsymbol{p}\boldsymbol{y}^*_j - \boldsymbol{w}\boldsymbol{x}^*_j\}. \tag{15}$$

where $(\boldsymbol{x}^*_j, \boldsymbol{y}^*_j) = (\boldsymbol{x}_j - TI_{CRS}(\boldsymbol{x}_j, \boldsymbol{y}_j; \boldsymbol{g}_{xj}, \boldsymbol{g}_{yj})\, \boldsymbol{g}_{xj}, \ \boldsymbol{y}_j + TI_{CRS}(\boldsymbol{x}_j, \boldsymbol{y}_j; \boldsymbol{g}_{xj}, \boldsymbol{g}_{yj})\, \boldsymbol{g}_{yj})$ is the technically efficient projection of observation j under $\widehat{T}_{CRS}$ in direction $(\boldsymbol{g}_{xj}, \boldsymbol{g}_{yj})$. Applying (15) to (7) and (8) yields $\widehat{T}_{PF}(.)$ and $\widehat{PI}(.)$ respectively. See Appendix A for the case of cost function.

The measures of technical, scale, and allocative inefficiencies ($\widehat{TI}$, $\widehat{SI}$, and $\widehat{AI}$) can be estimated as distances (3), (10), and (11) relative to frontier approximations (12), (13), and (7) respectively. While these estimates are consistent, more efficient estimators may be developed in an integrated approach to estimating the technology and these inefficiency concepts.

To this end, the current study proposes a weighted DEA (WDEA) approach. The key is to recognize that the conventional VRS technology $\widehat{T}_{VRS}$, the smallest feasible set meeting assumptions A1-A3, tends to underestimate T in a finite sample, and the subsequent underestimation of TI can be cast as systematic overestimation of SI and AI. Formally, consider WDEA technology $\widehat{T}_{W(\alpha,\beta)}$ defined as the weighted average of $\widehat{T}_{VRS}$, $\widehat{T}_{CRS}$, and $\widehat{T}_{PF}$ for given weights $\{1-\alpha,\ \alpha-\beta,\ \beta\}$ respectively;[3]

$$\begin{aligned}\widehat{T}_{W(\alpha,\beta)} &\equiv (1-\alpha)\widehat{T}_{VRS} + (\alpha-\beta)\widehat{T}_{CRS} + \beta\widehat{T}_{PF} \\ &= \widehat{T}_{VRS} + \alpha(\widehat{T}_{CRS} - \widehat{T}_{VRS}) + \beta(\widehat{T}_{PF} - \widehat{T}_{CRS}),\end{aligned} \tag{16}$$

which expands the conventional producible set $\widehat{T}_{VRS}$ by the α-portion of the input-output space conventionally regarded as SI and the β-portion of the space conventionally regarded as AI. In other words, the expansion of $\widehat{T}_{VRS}$ is implemented through partial transfers of the SI- and AI-space to the TI-space. The weights $\alpha,\ \beta \in [0,1]$ respectively generalize the extents of linear homogeneity and linear substitution assumptions used in technology approximation. Consequently, $\widehat{T}_{W(\alpha,\beta)}$ includes the conventional DEA frontiers of VRS, CRS, and PF as special cases; $\widehat{T}_{W(0,0)} = \widehat{T}_{VRS}$, $\widehat{T}_{W(1,0)} = \widehat{T}_{CRS}$, and $\widehat{T}_{W(1,1)} = \widehat{T}_{PF}$.

For decision $(\boldsymbol{x}_0, \boldsymbol{y}_0)$, let the TI measured under WDEA technology $\widehat{T}_{W(\alpha,\beta)}$ be

$$\begin{aligned}\widehat{TI}_{W(\alpha,\beta)}(\boldsymbol{x}_0, \boldsymbol{y}_0) &= (1-\alpha)\widehat{TI}_{VRS}(\boldsymbol{x}_0, \boldsymbol{y}_0) + (\alpha-\beta)\widehat{TI}_{CRS}(\boldsymbol{x}_0, \boldsymbol{y}_0) + \beta\widehat{PI}(\boldsymbol{x}_0, \boldsymbol{y}_0) \\ &= \widehat{TI}_{VRS}(\boldsymbol{x}_0, \boldsymbol{y}_0) + \alpha\widehat{SI}(\boldsymbol{x}_0, \boldsymbol{y}_0) + \beta\widehat{AI}(\boldsymbol{x}_0, \boldsymbol{y}_0),\end{aligned} \tag{17}$$

where $\widehat{TI}_{VRS}$, $\widehat{SI}$, and $\widehat{AI}$ are defined as (3), (10), and (11). With slight abuse of notations, let the associated SI and AI measures of $\widehat{T}_{W(\alpha,\beta)}$ be

$$\widehat{SI}_{W(\alpha,\beta)}(\boldsymbol{x}_0, \boldsymbol{y}_0) = \widehat{TI}_{CRS,W}(\boldsymbol{x}_0, \boldsymbol{y}_0) - \widehat{TI}_{W(\alpha,\beta)}(\boldsymbol{x}_0, \boldsymbol{y}_0), \tag{18}$$

$$\widehat{AI}_{W(\alpha,\beta)}(\boldsymbol{x}_0, \boldsymbol{y}_0) = \widehat{PI}_{W}(\boldsymbol{x}_0, \boldsymbol{y}_0) - \widehat{TI}_{CRS,W}(\boldsymbol{x}_0, \boldsymbol{y}_0) \tag{19}$$

where $\widehat{TI}_{CRS,W}(.)$ and $\widehat{PI}_{W}(.)$ are obtained by generating CRS and PF technologies around $\widehat{T}_{W(\alpha,\beta)}$ respectively.

In the case where the underestimation of TI is independent of the conventional measure of AI (i.e., $\beta = 0$) equation (18) reduces to;

$$\widehat{SI}_{W(\alpha,0)}(\boldsymbol{x}_0, \boldsymbol{y}_0) = (1-\alpha)(\widehat{TI}_{CRS}(\boldsymbol{x}_0, \boldsymbol{y}_0) - \widehat{TI}_{VRS}(\boldsymbol{x}_0, \boldsymbol{y}_0)) = (1-\alpha)\widehat{SI}(\boldsymbol{x}_0, \boldsymbol{y}_0), \tag{20}$$

[3] While notation $(\boldsymbol{w}, \boldsymbol{p})$ is omitted, $\widehat{T}_{W(\alpha,\beta)}$ clearly depends on the market prices through the profit frontier along $\widehat{T}_{PF}$.

and the exact transfer of the α-portion of conventional SI defines TI_{WDEA}. In a single-input single-output (x-y) space, figure 1 illustrates this case where the the postulated technological frontier lies (a solid-curve) between the CRS and VRS frontiers. The optimal projection of decision A to the VRS and CRS frontiers are shown at points B and C, yielding the conventional measures of TI and SI as distances AB and BC respectively. A postulated WDEA frontier is a weighted average of the VRS and CRS frontiers. The new TI and SI under WDEA are distances $AD(> AB)$ and $DC(< BC)$ where point D denotes the projection of point A onto the WDEA frontier.

Similarly, in the case where the technology is CRS (i.e., $\alpha = 1$), the equation (19) reduces to;

$$\widehat{AI}_{W(1,\beta)}(\boldsymbol{x}_0, \boldsymbol{y}_0) = (1-\beta)(\widehat{PI}(\boldsymbol{x}_0, \boldsymbol{y}_0) - \widehat{TI}_{CRS}(\boldsymbol{x}_0, \boldsymbol{y}_0)) = (1-\beta)\widehat{AI}(\boldsymbol{x}_0, \boldsymbol{y}_0), \tag{21}$$

and the exact transfer of the β-portion of conventional AI defines TI_{WDEA}. Figure 2 illustrates this situation, in which the postulated frontier is depicted between the CRS and cost frontiers.

The next part considers an optimal weight selection for α and β. Notations are simplified as follows: $TI_{T,i} \equiv TI_T(\boldsymbol{x}_i, \boldsymbol{y}_i)$, $AI_i \equiv AI(\boldsymbol{x}_i, \boldsymbol{y}_i)$, and $SI_i \equiv SI(\boldsymbol{x}_i, \boldsymbol{y}_i)$.

2.3 Weight Selection

Consider the following two-step weight selection process for α and β. The first step refines the VRS estimator at the observation level, say $\widehat{TI}_{T,i}$, by accounting for the finite-sample bias of DEA. The second step estimate optimal weights in the systematic relationships between the first-step estimate $\widehat{TI}_{T,i}$ and the conventional measures of TI, SI, and AI at the sample level.

Start with the second step. Consider optimal weights that minimize least square errors of the moment condition $E[TI_{T,i} - \widehat{TI}_{WDEA(\alpha,\beta),i}] = 0$. Under the definition of $\widehat{TI}_{W(\alpha,\beta),i}$ in (17), we have

$$\{\hat{\alpha}, \hat{\beta}\} = \underset{\alpha,\beta}{\text{argmin}} \left\{ \frac{1}{N} \sum_{i \in \mathbb{I}} \left(\widehat{TI}_{T,i} - (\widehat{TI}_{VRS,i} + \alpha \widehat{SI}_i + \beta \widehat{AI}_i) \right)^2 \right\}, \tag{22}$$

where $\widehat{TI}_{T,i} \approx TI_{T,i}$ is a consistent approximation obtained in the first step. The weight selection in (22) depends on the empirical distributions of $\widehat{TI}_{T,i}$, $\widehat{TI}_{VRS,i}$, $\widehat{SI}_i$, and $\widehat{AI}_i$ and is free of subjective judgements. Also, by construction the mean levels of $\widehat{TI}_{T,i}$ and $\widehat{TI}_{W(\alpha,\beta),i}$ are equated at the sample level. This ensures the consistency of $\widehat{TI}_{W(\alpha,\beta),i}$ without needing to impose explicit constraint $\alpha = \beta = 0$ as $N \to \infty$. Equation (22) thus admits the interpretation of WDEA as a refinement of the VRS estimator, while the WDEA and VRS estimators can be both consistent. The remainder of this section describes the derivation of $\widehat{TI}_{T,i}$ in the first step and discusses basic properties of the proposed WDEA estimator.

The conceptual underpinning of the first step draws on the subsample-bootstrap estimator proposed by Kneip, Simar, and Wilson (2008). The authors showed that for a convex technology, the behavior of the VRS estimator can be analyzed through the relative frequency of observations in a given neighborhood around the true frontier. Specifically, under a uniform density in the neighborhood, they derived an asymptotic distribution of the VRS estimator. Combined with the equivalence between the asymptotic properties of (additive) directional distance functions and those of (multiplicative) radial inefficiency measures (Simar, Vanhems, and Wilson, 2012), the $1-a$ confidence interval for $\widehat{TI}_{VRS,i}$ can be written as;

$$1 - a = Pr(C_a \leq \widehat{TI}_{VRS,i} - TI_T \leq C_b) \approx Pr(C_a \leq \widehat{TI}^*_{VRS,i} - \widehat{TI}_{VRS} \leq C_b) \tag{23}$$

where C_a and C_b represent lower and upper critical values for the deviation, and $\widehat{TI}^*_{VRS,i}$ is a bootstrap VRS estimator using $K(< N)$ observations sampled without replacement.[4] The critical values are substituted with their estimates $\widehat{C}_a = \psi_{a/2,K}$ and $\widehat{C}_b = \psi_{1-a/2,K}$ where $\psi_{x,K} \leq 0$ denotes the x-quantile of the bootstrap distribution $\{K^{2/(L+M+1)}(\widehat{TI}^{*,b}_{VRS,i} - \widehat{TI}_{VRS,i})\}_{b=1}^{B}$ from B bootstrap replications.

The concept behind the subsample-bootstrapping is that the distribution of the difference $\widehat{TI}_{VRS,i} - TI_T$ between the VRS estimator (in the sample) and the true value (in the universe) can be predicted from the distribution of the difference $\widehat{TI}^*_{VRS,i} - \widehat{TI}_{VRS}$ between the bootstrap-VRS estimator (in a subsample) and the VRS estimator (in the full sample) once the different rates of convergence under different sample sizes are accounted for. Then, the confidence interval in (23) can be estimated as

$$[\widehat{TI}_{VRS,i} - N^{-2/(L+M+1)}\psi_{1-a/2,K}, \quad \widehat{TI}_{VRS,i} - N^{-2/(L+M+1)}\psi_{a/2,K}], \tag{24}$$

which reflects the accuracy of local VRS estimator $\widehat{TI}_{VRS,i}$, predicted from the implicit sample density in the neighborhood. Let the mean of this confidence interval be referred to as mean bootstrap (MB) estimator (e.g., Simar, Vanhems, and Wilson, 2012), which makes upward ad-

[4] Another approach is to use the smooth-bootstrap method of Kneip, Simar, and Wilson (2011).

justments to the conventional $\widehat{TI}_{VRS,i}$;[5]

$$\widehat{TI}_{MB,i} = \widehat{TI}_{VRS,i} - \left(\frac{K}{N}\right)^{2/(L+M+1)} \frac{1}{B}\sum_{b=1}^{B}(\widehat{TI}^{*,b}_{VRS,i} - \widehat{TI}_{VRS,i}) \tag{25}$$

where $\widehat{TI}^{*,b}_{VRS,i} - \widehat{TI}_{VRS,i} \leq 0$.

To follow the conventional property of DEA calculations that at least one observation attains full efficiency, we adjust $\widehat{TI}_{MB,i}$ by a constant shift and define Adjusted MB (AMB) estimator;

$$\widehat{TI}_{AMB,i} = \widehat{TI}_{MB,i} - \bar{c} \tag{26}$$

where $\bar{c} = min\{\widehat{TI}_{MB,i} - \widehat{TI}_{VRS,i}\} \geq 0$.[6] [7]

Now refine technology approximation $\widehat{T}_{VRS}$ by collectively utilizing $\widehat{TI}_{AMB,i}$ at the sample level. The new technology approximation Adjusted VRS (AVRS) is given as;

$$\begin{aligned} \widehat{T}_{AVRS} = \{(\boldsymbol{x}',\ \boldsymbol{y}') :\ & \sum_{j\in I}\lambda_j(y_j + \widehat{TI}_{AMB,j}\,\boldsymbol{g}_{yj}))+ \geq \boldsymbol{y}', \\ & \sum_{j\in I}\lambda_j(\boldsymbol{x}_j - \widehat{TI}_{AMB,j}\,\boldsymbol{g}_{xj}) \leq \boldsymbol{x}',\ \sum_{j\in I}\lambda_j = 1,\ \boldsymbol{\lambda}\in\mathbb{R}^N_+\}, \end{aligned} \tag{27}$$

yielding the associated inefficiency, $\widehat{TI}_{AVRS,i} = \max\{b : (\boldsymbol{x}_i - b\boldsymbol{g}_{xi}, \boldsymbol{y}_i + b\boldsymbol{g}_{yi}) \in \widehat{T}_{AVRS}\}$. Note that the magnitude of the constant $\bar{c}$ in (26) directly affects the mean TI under AVRS and WDEA.

In short, the proposed weight selection proceeds by constructing $\widehat{TI}_{T,i}$ by $\widehat{TI}_{AVRS,i}$ and estimating optimal weights $\hat{\alpha}$ and $\hat{\beta}$ by equation (22). The weights capture the sample correlations between the bias-correction $\widehat{TI}_{AVRS,i} - \widehat{TI}_{VRS,i}$ and the conventional measures of scale and allocative inefficiency, $\widehat{SI}_i$ and $\widehat{AI}_i$.

To derive basic properties of the WDEA estimator, consider the following relationships be-

[5]The mean can be replaced with the median or mode of distribution $\{K^{2/(L+M+1)}(\widehat{TI}^{*,b}_{VRS,} - \widehat{TI}_{VRS,i})\}^B_{b=1}$. Simulation study may be helpful to investigate these alternative estimators.

[6]The cases where the MB estimator is not available (e.g., the observation with the largest output under input-oriented efficiency) are removed from the calculation of $\bar{c}$.

[7]For simplicity, the minimum difference is used in this study. If the mean, instead of the minimum, of the difference is used, the bias-corrected TI estimator needs to be artificially adjusted to be at least as large as the VRS estimator.

tween the unobserved true $TI_{T,i}$ and its estimators by VRS, AVRS, and WDEA;

$$\begin{aligned} VRS: \ & TI_{T,i} = \widehat{TI}_{VRS,i} + \varepsilon_{VRS,i}, \quad \varepsilon_{VRS,i} > 0 \\ AVRS: \ & TI_{T,i} = \widehat{TI}_{AVRS,i} + \bar{c} + \varepsilon_{AVRS,i}, \quad E[\varepsilon_{AVRS,i}] = 0 \\ WDEA: \ & TI_{T,i} = \widehat{TI}_{VRS,i} + \alpha\widehat{SI}_{VRS,i} + \beta\widehat{AI}_{CRS,i} + \bar{c} + \varepsilon_{W(\alpha,\beta),i}, \quad E[\varepsilon_{W(\alpha,\beta),i}] = 0 \end{aligned} \tag{28}$$

where $\varepsilon_{VRS,i}$, $\varepsilon_{AVRS,i}$, and $\varepsilon_{W(\alpha,\beta),i}$ are the residual terms that close these identities. In the first equation, the well-known one-sided bias of the VRS estimator (i.e. $\varepsilon_{VRS,i} > 0$) implies mean-inconsistency $E[TI_{T,i} - \widehat{TI}_{VRS,i}] = E[\varepsilon_{VRS,i}] > 0$, while it is asymptotically consistent in that $E[TI_{T,i} - \widehat{TI}_{VRS,i}] \to 0$ for a sufficiently large sample (Banker, Gadh, and Gorr, 1993). In the second equation, the AVRS estimator with constant $\bar{c}$ is assumed to be consistent, given the properties of the bias correction method. With this assumption, combining the second and the third equations to eliminate $TI_{T,i}$ and using $\hat{\alpha}$ and $\hat{\beta}$ in (22) yields a condition for consistent WDEA estimator.

Remark 1. *In (22) and (28), if $E[\varepsilon_{AVRS,i}|\widehat{TI}_{AVRS,i}, \widehat{TI}_{VRS,i}, \widehat{SI}_i, \widehat{AI}_i] = 0$, then the WDEA estimator is consistent, or $E[\varepsilon_{W(\hat{\alpha},\hat{\beta}),i}] = 0$.*

Simple comparisons for estimation efficiency can be made;

Remark 2. *In (28), if $E[\varepsilon_{AVRS,i}|\widehat{TI}_{AVRS,i}, \widehat{TI}_{VRS,i}] = 0$, then the AVRS estimator is more efficient than the VRS estimator in that $E[(\varepsilon_{AVRS,i})^2] \leq E[(\varepsilon_{VRS,i})^2]$ where $\varepsilon_{VRS,i} = (\widehat{TI}_{AVRS,i} - \widehat{TI}_{VRS,i}) + \bar{c} + \varepsilon_{AVRS,i}$.*

Remark 3. *In (22) and (28), if $E[\varepsilon_{W(\hat{\alpha},\hat{\beta}),i}|\widehat{SI}_i, \widehat{AI}_i] = 0$ and $\hat{\alpha}$, $\hat{\beta} \geq 0$, then the WDEA estimator is more efficient than the VRS estimator in that $E[(\varepsilon_{W(\hat{\alpha},\hat{\beta}),i})^2] \leq E[(\varepsilon_{VRS,i})^2]$ where $\varepsilon_{VRS,i} = \alpha\widehat{SI}_i + \beta\widehat{AI}_i + \bar{c} + \varepsilon_{W(\hat{\alpha},\hat{\beta}),i}$.*

Remark 4. *In (22) and (28), if $E[\varepsilon_{AVRS,i}|\widehat{TI}_{AVRS,i}, \widehat{TI}_{VRS,i}, \widehat{SI}_i, \widehat{AI}_i] = 0$ and $\hat{\alpha}$, $\hat{\beta} \leq 0$, then the AVRS estimator is more efficient than the WDEA estimator in that $E[(\varepsilon_{AVRS,i})^2] \leq E[(\varepsilon_{W(\hat{\alpha},\hat{\beta}),i})^2]$ where $\varepsilon_{AVRS,i} = \hat{\alpha}\widehat{SI}_i + \hat{\beta}\widehat{AI}_i + (\widehat{TI}_{VRS,i} - \widehat{TI}_{AVRS,i}) + \varepsilon_{W(\hat{\alpha},\hat{\beta}),i}$.*

Remark 2 follows from $\widehat{TI}_{AVRS,i} - \widehat{TI}_{VRS,i} \geq 0$ and $\bar{c} \geq 0$. Remark 3 similarly follows from $\hat{\alpha}$, $\hat{\beta}$, $\bar{c} \geq 0$. Remark 4 states that under $\hat{\alpha}$, $\hat{\beta} \leq 0$, incorporating AI and SI into technology estimation would be counterproductive. Meanwhile, there seems no simple condition under which the WDEA estimator is more efficient than the AVRS counterpart.

3 Application

3.1 Data

We demonstrate the proposed methodology with an application to U.S. dairy production using the 2010 USDA Agricultural Resource Management Survey (ARMS) Phase III dairy version. The data set drawn from 26 states of prominent dairy operations, represents about 90% of U.S. milk production and contains output quantities, revenues, and expense of dairy production, as well as the estimated cost-of-production variables by the Economic Research Service (USDA-ERS) such as capital recovery cost, homegrown feed production cost, and the opportunity cost of grazing pasture. Barring operations with less than 50 milking cows and those producing organic milk, as well as clear outliers in terms of the unit cost of production,[8] we use the total of 1,005 observations in the following analysis. While certain segments of the population are over-sampled in the ARMS, for simplicity we make no statistical adjustments in this study.

The sample is split into five regions by state and separately analyzed by region. The five regions are: Northwest (CO, WA, OR, ID), Southwest (CA, NM, AZ, TX), Midwest (WI, MN, MI, IL, IA, MO, ID, KS, OH), Northeast (NY, PA, ME, VT, VA), and Southeast (KT, TN, GA, FL). These regions differ in their climate and regional market conditions. Also, the dairy operations in the Northwest and Southwest tend to be larger and rely more on purchased feed, while those in the Midwest and Northeast on average produce about a half of forage on their own (e.g., see the lower panel of Table 1).

We use a single-output, six-input specification for dairy production. The output is the total revenue of dairy operation, and the inputs are the number of milking cows, the estimated cost of homegrown feed (including the estimated cost of grazing), the cost of purchased feed, the total labor hours (i.e. the total working hours of owner-operators and hired labor), the sum of non-feed operating expenses, and the estimated capital recovery cost. Summary statistics are provided in Table 1. Given the cross-sectional nature of the study, the revenue and expense variables are not converted into quasi-quantity measures. Uniform factor prices are applied across geographical regions; the prices for the monetary variables are set to one, while the input prices of cow and labor hour are set to \$350 and \$11.8 respectively, based on the estimated rental cost of a cow and the average labor wage rate in 2010.[9]

[8] We define outliers by regressing unit cost (i.e., cost per hundredweight of milk and cost per cow) on a set of variables (i.e., $\boldsymbol{w}_i$ in equation (29) below) and identifying observations with Cooks distance greater than 4/N or studentised residual greater than 2 in absolute value.

[9] The rental rate of a cow is estimated as the sum of the net milking income (\$400), the average value of a calf (\$250: the average of male and female calves), and the depreciation (-\$300).

3.2 Efficiency Estimation

Table 2 reports the summary of the estimated efficiency scores by region, including three versions of technical efficiency ($TE = 1 - TI$) under VRS, AVRS, and WDEA models, as well as scale efficiency ($SE = 1 - SI$) and allocative efficiency ($AE = 1 - AI$). At the mean, the TE-VRS ranges from 0.779 to 0.981 across five regions, which is about 13 to 21 percentage-point higher than the TE-AVRS (0.559 to 0.753) and the TE-WDEA (0.571 to 0.757). In all three measures, the estimated TE is relatively high for the Northwest and Northeast, in which many producers operate large-scale dairies using relatively modern facilities. The Southeast has the lowest average TE in all three measures. A common perception is that the challenge of operating under high temperatures and humidity, which negatively affect animal health and milk quality, leads to prevalent inefficiencies in this region.

The mean levels of SE and AE range from 0.837 to 0.910 and 0.624 to 0.721 respectively. Scale efficiency is relatively high in the Northwest and Southwest where producers are predominantly large and also in the Northeast where large-scale producers are very rare. Scale inefficiency is more prevalent in the Midwest and Southeast, the regions with diverse operational scales. On the other hand, the distribution of allocative efficiency is similar across regions, and much of the inefficient resource allocation seems to be explained by the suboptimal herd size relative to other inputs (as indicated by the high sensitivity of AE to the price increase for cow). With the expansion of large-scale dairies and their tendency to reduce capital- and labor-inputs per animal, many producers of smaller scales are deemed increasingly underinvested in their stock of animals and over-invested in other inputs.

The estimation of optimal weights for WDEA suggests that a 14.7 to 42.6% of the conventional measure of AI and a 27.3 to 49.4% of the conventional measure of AI can be attributed to the underestimated TI (Table 3). We note that the optimal weights, currently estimated through a regression without a constant term, appear relatively sensitive to constant $\bar{c}$, and $\bar{c}$ itself may be influenced by outliers in the data. This aspect may merit further investigation in future research.

Figures 3 - 7 illustrate the distributions of the three versions of TE estimates in the form of kernel density estimate on domain [0, 1]. The estimation without distributional assumptions on TE is a major advantage of DEA over other techniques like SFA. However, extreme anomalies in the distribution of estimated TE may indicate the possibility of misspecification or a systematic bias in technology estimation. In the current application, the distribution of TE-VRS exhibits multiple modes (for the Midwest and Southeast in particular) and large skewness (for the Northwest and Northeast). In comparison, the distribution of TE-AVRS is smoother and more symmetric thanks to the finite-sample bias-correction. The distribution of TE-WDEA is centered around the mean of TE-AVRS and relatively symmetric around the mean, while showing some traits of the distribution of TE-VRS.

3.3 Performance Comparison

We examine the relative performance of the VRS, AVRS, and WDEA estimators using the analogy of technical efficiency to unobserved "managerial ability," a common interpretation for varying efficiency in empirical analysis. For a given measure of management benchmarking metrics r_i, consider

$$r_i = \gamma\, \widehat{TE}_i + \boldsymbol{w}_i \boldsymbol{\delta} + \varepsilon_i \tag{29}$$

where $\boldsymbol{w}_i$ is a set of variables, including information about factor prices and operational scale, $\widehat{TE}_i$ a proxy for managerial ability, and ε_i an error term. This allows us to compare the explanatory power of $\widehat{TE}_{m,i} = 1 - \widehat{TI}_{m,i}$ across models $m \in \{$ VRS, AVRS, WDEA$\}$, while controlling for the reduced-form relationships between r_i and $\boldsymbol{w}_i$. Neither the AVRS nor WDEA estimator is guaranteed to be a better proxy for managerial ability than the VRS estimator since the additional structures of those models may introduce more noise than useful information, which increases attenuation bias in $\widehat{TE}_i$.

The choice of variables for (29) requires some caution. Under the input-oriented TE measurement, the dependent variable r_i cannot be a function of the total cost or cost inefficiency. It is because $\widehat{TE}_{WDEA,i}$ is by construction more strongly correlated with the cost than $\widehat{TE}_{VRS,i}$ is. Also, the two remaining components of cost inefficiency, SE and AE, may not be used as covariates since they are likely to confound the analogy of TE to managerial ability. In particular, it is unclear how AI, a catch-all residual explanation of cost inefficiency net TI and SI, relates to managerial ability and dependent variable r_i. Instead of SE and AE, we include herd size and state-average milk price and feed cost per milk output in covariates $\boldsymbol{w}_i$.

Our choice for r_i is a partial measure of profit, as well as revenue-based returns to production assets. Specifically, we utilize income over feed cost (IOFC: milk revenue minus feed cost, a common benchmarking metric in the U.S. dairy industry) per animal and per output, as well as the revenue per animal and per asset dollar (the latter is known as asset turnover rate). The correlations with these conventional benchmarking metrics help assess empirical relevance of the VRS, AVRS, and WDEA estimators in our context.

Table 4 reports the regression results for IOFC for the Midwest region, or the region with the largest sample size. The results for other regions are qualitatively similar. The coefficient estimates indicate that one-percentage point increase in TE predicts an increase in IOFC per hundredweight (cwt) of milk by \$0.08 under VRS, \$0.09 under AVRS, and \$0.15 under WDEA respectively (columns (1)-(3)). Similarly, one-percentage point increase in TE indicates an increase in IOFC per cow by \$17.20 under VRS, \$25.82 under AVRS, and \$34.44 under WDEA respectively (columns (4)-(6)). Thus, in this case the higher magnitude of WDEA coefficient than other two models, or the higher explanatory power for IOFC, appears to suggest the relative superiority of WDEA. Numerically, the results under WDEA imply that a difference in TE by one-standard deviation (12.9 percentage points) translates into \$1.94 difference in IOFC per cwt, compared to the milk price of \$16.5 at that time, and \$444 difference in IOFC per cow, or about \$94,000 difference at the mean herd size of 211. In table 5, we report the parallel estimation results for revenue per cow and revenue per asset.

Table 6 summarizes the estimated coefficients of the three TE measures by region and dependent variable. The above results for the Midwest are a fair representation of the results for other four regions. In most cases, the WDEA estimator shows a substantially higher explanatory power than the VRS estimator, whereas the AVRS estimator does not appear to systematically improve the VRS model. On average, the coefficient estimate under WDEA is 80% and 72% higher than those under VRS and AVRS respectively.

4 Conclusions

Benchmarking technical efficiency has been a major area of research in applied production theory. The linkage among theoretical concepts of technical, scale, and allocative efficiencies offers an opportunity to improve empirical analysis. The proposed Weighted DEA (WDEA) extends the standard VRS technology approximation by accounting for the correlations between its finite-sample bias and the conventional measures of scale and allocative inefficiencies. The new frontier approximation is a weighted average of the VRS, CRS, and profit frontiers where the weights are cast as the optimal degrees of linear homogeneity and linear substitutability to be incorporated.

In the application to U.S. dairy operations data, the estimated technical efficiency (TE) is 13 to 21 percentage-point higher under WDEA than VRS, depending on the production region. The estimated TE under WDEA exhibits less anomalies in its distribution than the estimate under VRS. It is also more strongly correlated with common management metrics in the dairy industry, indicating improvements over the conventional approach. The proposed approach may be particularly useful in cases of large heterogeneity in data, under which both SFA and DEA tend to perform poorly. Further studies are needed to refine the weight selection rule and increase the understanding of empirical properties and performance of the new estimator.

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5 Tables and Figures

Table 1: Summary of Production Variables

A. All Regions	Distributional properties						
	Mean	S.D.	5th	25th	50th	75th	95th
Dairy Revenue ($1,000)	1,484	3,106	137	246	436	1,169	6,493
Milking Cows (animals)	416	843	55	84	140	340	1,725
Purchased Feed ($1,000)	597	1,417	20	63	130	426	2,909
Homegrown Feed ($1,000)	231	445	0	44	102	228	869
Labor (1,000 hours)	15	30	3	5	7	12	48
Other Operation Cost ($1,000)	235	418	23	48	84	205	1,036
Capital Recovery Cost ($1,000)	227	560	32	62	101	195	741

B. By Region	Northwest		Southwest		Midwest		Northeast		Southeast	
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
Dairy Revenue ($1,000)	2,315	3,692	4,073	4,771	795	1,410	680	953	861	1845
Milking Cows (animals)	674	1068	1177	1388	211	331	180	219	248	446
Purchased Feed ($1,000)	966	1,485	1,886	2,388	234	543	207	291	345	842
Homegrown Feed ($1,000)	289	495	376	690	225	425	205	315	130	283
Labor (1,000 hours)	23	29	30	51	9	9	11	20	14	45
Other Operation Cost ($1,000)	363	481	576	682	145	221	156	235	126	240
Capital Recovery Cost ($1,000)	294	454	438	571	142	184	147	187	157	263

Data source: USDA-ARMS 2010 Dairy Costs and Returns Report. The total of 1006 observations, excluding organic dairies, dairies with less than 50 cows, and production cost outliers. Five regions are defined as: Northwest (CO, WA, OR, ID: N=138), Southwest (CA, NM, AZ, TX: N=135), Midwest (WI, MN, MI, IL, IA, MO, ID, KS, OH: N=368), Northeast (NY, PA, ME, VT, VA: N=154), and Southeast (KT, TN, GA, FL: N=210).

Table 2: Summary of Estimated Efficiencies

	Mean	S.D.	Min	25th	50th	75th	Max
Northwest							
TE (VRS)	0.891	0.119	0.479	0.823	0.921	1.000	1.000
SE	0.907	0.115	0.533	0.857	0.957	0.995	1.000
AE	0.644	0.107	0.365	0.575	0.652	0.714	1.000
TE (AVRS)	0.753	0.108	0.479	0.683	0.755	0.814	1.000
TE (WDEA)	0.757	0.105	0.424	0.688	0.776	0.838	1.000
Southwest							
TE (VRS)	0.830	0.164	0.439	0.701	0.851	1.000	1.000
SE	0.908	0.119	0.414	0.864	0.965	0.992	1.000
AE	0.657	0.125	0.257	0.596	0.675	0.733	1.000
TE (AVRS)	0.667	0.159	0.369	0.535	0.639	0.787	1.000
TE (WDEA)	0.676	0.137	0.367	0.584	0.692	0.803	1.000
Midwest							
TE (VRS)	0.797	0.149	0.387	0.682	0.793	0.922	1.000
SE	0.842	0.172	0.276	0.750	0.899	0.989	1.000
AE	0.690	0.114	0.199	0.624	0.703	0.767	1.000
TE (AVRS)	0.663	0.115	0.368	0.580	0.654	0.743	1.000
TE (WDEA)	0.665	0.120	0.319	0.574	0.665	0.760	1.000
Northeast							
TE (VRS)	0.878	0.140	0.414	0.769	0.937	1.000	1.000
SE	0.898	0.122	0.506	0.837	0.944	0.997	1.000
AE	0.674	0.126	0.256	0.604	0.683	0.760	1.000
TE (AVRS)	0.706	0.133	0.377	0.610	0.709	0.805	1.000
TE (WDEA)	0.714	0.111	0.360	0.634	0.742	0.801	1.000
Southeast							
TE (VRS)	0.779	0.172	0.414	0.634	0.765	0.994	1.000
SE	0.839	0.157	0.362	0.754	0.886	0.975	1.000
AE	0.718	0.134	0.154	0.642	0.757	0.815	1.000
TE (AVRS)	0.559	0.140	0.326	0.455	0.523	0.646	1.000
TE (WDEA)	0.571	0.129	0.306	0.467	0.570	0.664	1.000

Three technical efficiency (TE=1=TI) measures are obtained under VRS, AVRS, and WDEA models by equations (3), (**??**), and (17). SE(=1-SI) and AE(=1-AI) are scale and allocative efficiency estimates respectively by equations (10), and (11).

Table 3: Optimal Weight Estimation

	Northwest	Southwest	Midwest	Northeast	Southeast
Scale Inefficiency	0.147***	0.197***	0.297***	0.258***	0.426***
	(0.045)	(0.052)	(0.016)	(0.044)	(0.030)
Allocative Inefficiency	0.336***	0.397***	0.273***	0.422***	0.494***
	(0.018)	(0.021)	(0.011)	(0.020)	(0.021)
Adj. R Squared	0.814	0.801	0.843	0.836	0.875
Observations	138	135	368	154	210

1. Statistical significance: *** $\alpha = 0.01$, ** $\alpha = 0.05$, * $\alpha = 0.1$. Standard errors in parentheses.

2. OLS estimation without constant term based on equation (22) for dependent variable $\widehat{TI}_{AVRS,i} - \widehat{TI}_{VRS,i}$.

Table 4: Regression of IOFC on Technical Efficiency, Midwest

	IOFC per cwt of milk			IOFC per cow		
Variable	(1)	(2)	(3)	(4)	(5)	(6)
TE (VRS)	0.075***			17.207***		
	(0.014)			(2.893)		
TE (AVRS)		0.089***			23.815***	
		(0.019)			(3.840)	
TE (WDEA)			0.149***			34.444***
			(0.017)			(3.409)
Cow (100 animals)	0.142**	0.109*	0.059	53.568***	42.857***	34.262***
	(0.063)	(0.065)	(0.061)	(13.186)	(13.472)	(12.450)
Avg.Milk price ($/cwt)	0.026	0.036	-0.075	49.348	50.435	25.811
	(0.346)	(0.349)	(0.326)	(72.261)	(71.976)	(66.939)
Avg. Feed cost ($/cwt)	-0.588	-0.610	-0.658	-14.608	-19.679	-30.703
	(0.457)	(0.461)	(0.430)	(95.515)	(95.142)	(88.402)
Constant	4.592	4.837	3.297	-1137.467	-1286.057	-1453.617
	(9.246)	(9.336)	(8.682)	(1930.580)	(1924.915)	(1782.969)
Adj. R Squared	0.095	0.079	0.198	0.139	0.145	0.262

Statistical significance: *** $\alpha = 0.01$, ** $\alpha = 0.05$, * $\alpha = 0.1$. Standard errors in parentheses. Estimation based on equation (29) with 368 observations.

Table 5: Regression of Revenue on Technical Efficiency, Midwest

	revenue per cow			revenue per asset		
Variable	(1)	(2)	(3)	(4)	(5)	(6)
TE (VRS)	10.364***			0.076		
	(3.021)			(0.073)		
TE (AVRS)		26.384***			0.172*	
		(3.847)			(0.097)	
TE (WDEA)			23.041***			0.256***
			(3.715)			(0.092)
Cow (100 animals)	70.549***	53.016***	56.887***	3.059***	2.950***	2.881***
	(13.767)	(13.499)	(13.570)	(0.332)	(0.339)	(0.335)
Avg.Milk price ($/cwt)	324.638***	320.994***	308.254***	1.690	1.670	1.486
	(75.440)	(72.119)	(72.958)	(1.817)	(1.812)	(1.803)
Avg. Feed cost ($/cwt)	379.580***	376.142***	369.100***	0.315	0.290	0.208
	(99.717)	(95.331)	(96.351)	(2.402)	(2.395)	(2.381)
Constant	-7087.192***	-7875.655***	-7380.962***	-16.232	-20.822	-22.344
	(2015.520)	(1928.726)	(1943.286)	(48.557)	(48.463)	(48.014)
Adj. R Squared	0.157	0.230	0.213	0.201	0.206	0.215

Statistical significance: *** $\alpha = 0.01$, ** $\alpha = 0.05$, * $\alpha = 0.1$. Standard errors in parentheses. Estimation based on equation (29) with 368 observations.

Table 6: Summary of Coefficient Estimates of Technical Efficiency

	Northwest	Southwest	Midwest	Northeast	Southeast
A. Income Over Feed Cost ($/cwt)					
TE (VRS)	0.109***	0.114***	0.075***	0.076***	0.119***
TE (AVRS)	0.094***	0.117***	0.089***	0.046*	0.114***
TE (WDEA)	0.178***	0.172***	0.149***	0.158***	0.212***
B. Income Over Feed Cost ($/cow)					
TE (VRS)	22.444***	24.304***	17.207***	16.334***	24.002***
TE (AVRS)	18.066***	24.490***	23.815***	11.474**	21.812***
TE (WDEA)	37.665***	36.592***	34.444***	33.821***	46.989***
C. Revenue per cow ($/cow)					
TE (VRS)	16.592***	20.948***	10.364***	17.388***	16.514***
TE (AVRS)	13.588**	18.589***	26.384***	16.128***	15.470***
TE (WDEA)	22.448***	28.244***	23.041***	33.013***	35.295***
D. Revenue per asset (asset turnover)					
TE (VRS)	0.581	0.719***	0.076	0.179	0.197*
TE (AVRS)	0.001	0.945***	0.172*	0.376**	0.477***
TE (WDEA)	0.698	0.870***	0.256***	0.227	0.360**

The table contains estimated coefficients of TE from the regression models used for tables 4 and 5. Estimation for dependent variables A (Income Over Feed Cost) through D (Revenue per asset), based on equation (29).

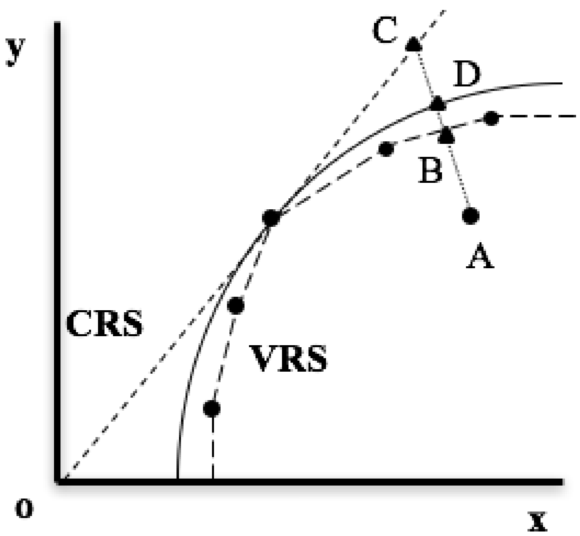

Figure 1: CRS, VRS, and Postulated Frontiers

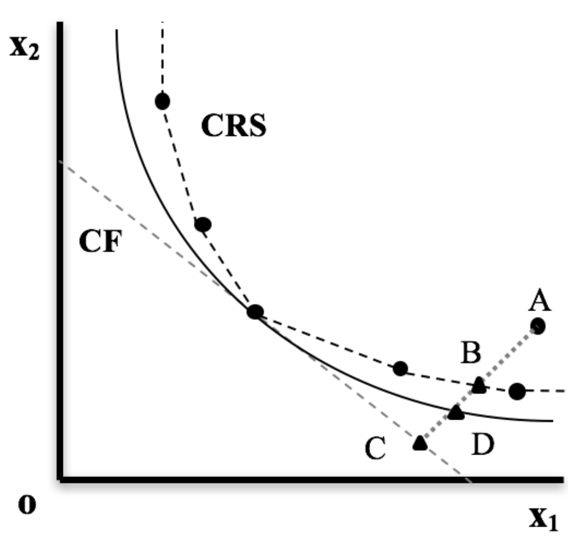

Figure 2: Cost, CRS, and Postulated Frontiers

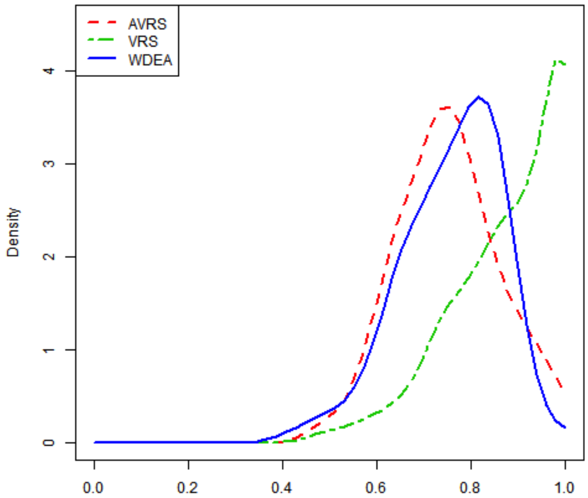

Figure 3: Kernel Density of TE, Northwest

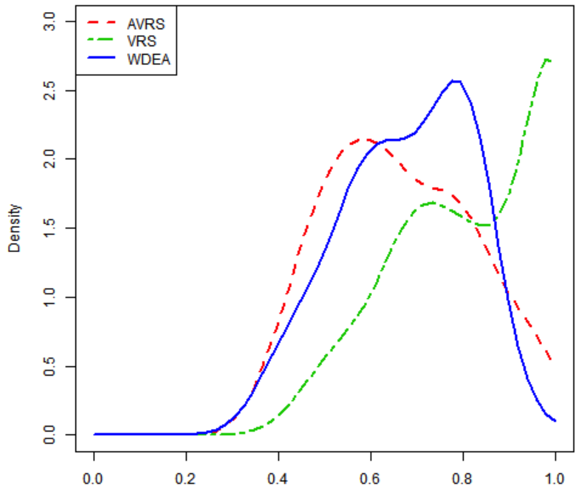

Figure 4: Kernel Density of TE, Southwest

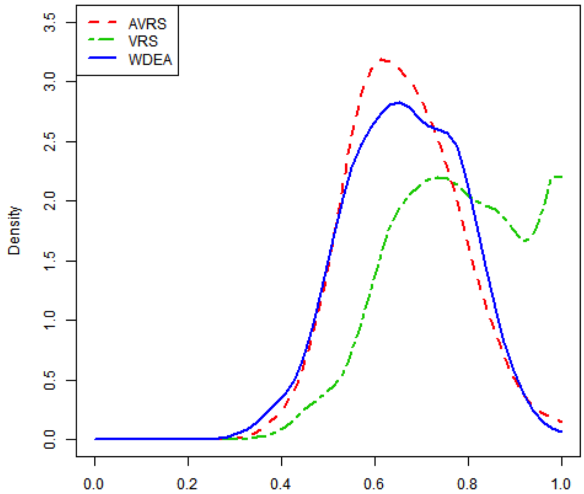

Figure 5: Kernel Density of TE, Midwest

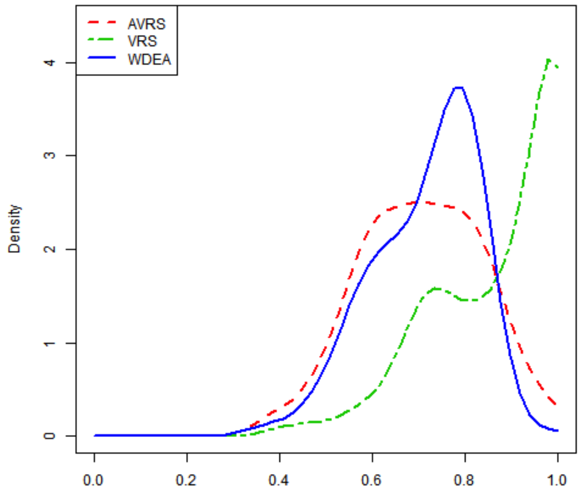

Figure 6: Kernel Density of TE, Northeast

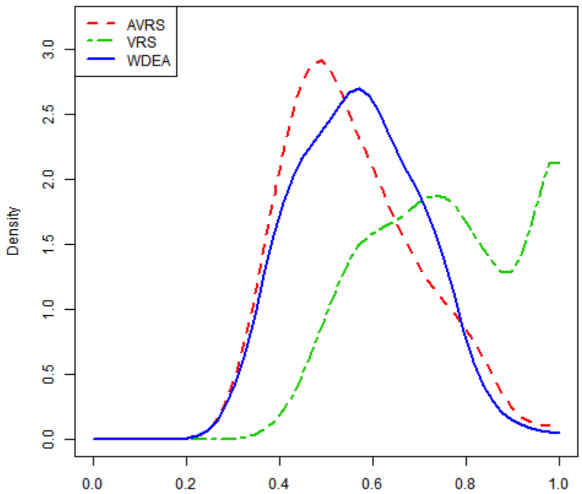

Figure 7: Kernel Density of TE, Southeast

A Input-oriented Efficiency

For given observation $(\boldsymbol{x}_0, \boldsymbol{y}_0)$, input-orientation efficiency refers to the evaluation along the radial contraction of inputs $\boldsymbol{x}_0$ for given outputs $\boldsymbol{y}_0$. This is equivalent to setting the directional distance function in the direction $(\boldsymbol{g}_x, \boldsymbol{g}_y) = (\boldsymbol{x}_0, \boldsymbol{0})$. Denote this input-oriented function $TI_T^I(\boldsymbol{x}_0, \boldsymbol{y}_0) = TI_T(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{x}_0, \boldsymbol{0})$, which becomes

$$TI_T^I(\boldsymbol{x}_0, \boldsymbol{y}_0) = \max\{b \in \mathbb{R} : \ (\boldsymbol{x}_0 - b\boldsymbol{x}_0) = (1-b)\, \boldsymbol{x}_0 \in V_T(\boldsymbol{y}_0)\}. \tag{A.1}$$

The dual cost function is given by;

$$\begin{aligned} C_T(\boldsymbol{w}, \boldsymbol{y}_0) &= \min_{\boldsymbol{x}}\{\boldsymbol{wx} : \ \boldsymbol{x} \in V_T(\boldsymbol{y}_0)\} \\ &= \min_{\boldsymbol{x}}\{\boldsymbol{wx} - TI_T^I(\boldsymbol{x}, \boldsymbol{y}_0)\, (\boldsymbol{wx})\} \\ &= \min_{\boldsymbol{x}}\{\boldsymbol{wx}\, (1 - TI_T^I(\boldsymbol{x}, \boldsymbol{y}_0))\}. \end{aligned} \tag{A.2}$$

By duality, we have

$$TI: \quad TI_T^I(\boldsymbol{x}_0, \boldsymbol{y}_0) = \max_{\boldsymbol{w}} \left\{ \frac{\boldsymbol{wx}_0 - C_T(\boldsymbol{w}, \boldsymbol{y}_0)}{\boldsymbol{wx}_0} \right\} \tag{A.3}$$

Cost inefficiency (CI) at given input price $\boldsymbol{w}$ is

$$CI: \quad TI_{CF}^I(\boldsymbol{x}_0, \boldsymbol{y}_0, \boldsymbol{w}) = \frac{\boldsymbol{wx}_0 - C_{CRS}(\boldsymbol{w}, \boldsymbol{y}_0)}{\boldsymbol{wx}_0} \tag{A.4}$$

where $C_{CF}(\boldsymbol{w}, \boldsymbol{y}_0)$ is the cost function associated with CRS input set $V_{CRS}(\boldsymbol{y}_0)$. CI can be obtained by replacing $C_{CF}(\boldsymbol{w}, \boldsymbol{y}_0)$ with its estimate;

$$\widehat{C}_{CF(\boldsymbol{w}, \boldsymbol{y}_0)} = \min_{\boldsymbol{x}}\{\boldsymbol{wx} : \ \boldsymbol{x} \in \widehat{V}_{CRS}(\boldsymbol{y}_0)\} \tag{A.5}$$

$$= \min\{\theta : \ \sum_j \lambda_j \boldsymbol{y}_j \geq \boldsymbol{y}_0, \ \ \boldsymbol{w}(\sum_j \lambda_j \boldsymbol{x}_j) \leq \theta\}, \tag{A.6}$$

which attains the minimum cost at given output level $\boldsymbol{y}_0$. The additive decomposition parallel to (9) becomes;

$$\begin{aligned} CI &= TI + SI + AI \\ & TI_T^I + (TI_{CRS}^I - TI_T^I) + (CI - TI_{CRS}^I). \end{aligned} \tag{A.7}$$