The time limit for this exam is four hours. The exam has four sections. Each section includes two questions. Answer one question from each section. Before turning in your exam, number each page of your answers in sequential order, and identify the question (for example, III.2 for section III and question 2). Indicate, by circling below, the questions you completed. If you answer more than one question in a section and do not circle a question corresponding to that section, the first of the two that appears in your solution paper will be graded.

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STUDENT ID LETTER: _______ (Fill in your code letter)

Answer one question from each section. Indicate the one you answered by circling:

Section I: Question 1 Question 2
Section II: Question 1 Question 2
Section III: Question 1 Question 2
Section IV: Question 1 Question 2

*** TURN IN THIS SHEET WITH YOUR ANSWER PAGES ***
Part I

Answer at most one question from Part I.
Question I.1

All of this question is concerned with the following indirect utility function:

\[ v = \alpha_0 + \alpha_1 \log \left( \frac{p_1}{w} \right) + \alpha_2 \log \left( \frac{p_2}{w} \right) + \beta_{11} \left[ \log \left( \frac{p_1}{w} \right) \right]^2 + \beta_{22} \left[ \log \left( \frac{p_2}{w} \right) \right]^2 + \beta_{12} \log \left( \frac{p_1}{w} \right) \log \left( \frac{p_2}{w} \right), \]

where “log” is the natural log, \( p_1 \) is the price of the good \( x_1 \), \( p_2 \) is the price of the good \( x_2 \), \( w \) is wealth, and the \( \alpha \)’s and \( \beta \)’s are parameters. Assume that both prices and \( w \) are strictly greater than 0.

(a) What property should any indirect utility function have with respect to changes in prices? Does this indirect utility function satisfy this property? Do you need to impose any constraints on the \( \alpha \) and/or \( \beta \) parameters to ensure that this indirect utility function satisfies this restriction?

(b) What property should any indirect utility function have with respect to a change in wealth? Does the above indirect utility function satisfy this property? Do you need to impose any constraints on the \( \alpha \) and/or \( \beta \) parameters to ensure that this indirect utility function satisfies this restriction?

(c) Derive the Walrasian demand for goods \( x_1 \) and \( x_2 \). Do these demands satisfy Walras’ law and the homogeneity property required of Walrasian demand functions?

(d) Duality properties of consumer demand imply that the expenditure function can be obtained from the indirect utility function by using some algebra. However, to do so you will need to use the quadratic formula. Suppose that you cannot remember the quadratic formula. What other method can you use based on duality properties to obtain the expenditure function? You do not have to do it, but just explain how to do it. Hint: Assume that you are really good at solving (partial) differential equations, even though you do not remember the quadratic formula.
Question I.2

(Risk, Expected Utility and Insurance.) Consider an individual who faces a risky environment and has a Bernoulli utility function $u(x)$, where $x$ is an amount of money, and $u(x)$ is strictly concave. This person has a wealth $w$, and faces risk in the form that he or she will lose $D$ dollars with probability $\pi$. This individual wants to buy insurance to reduce this risk. As in class, insurance is available in the following form. A person can buy $\alpha$ insurance claims, each of which pays a value of $1$ if the loss occurs. The price of these claims is $q$.

(a) Express this person’s expected utility if he or she buys $\alpha$ insurance claims. Show the first-order condition for this person to maximize his or her expected utility.

(b) Assume that the person buys at least some insurance, so that the first-order condition is an equality. Suppose that the price of insurance is NOT actuarially fair, which implies that $q > \pi$. Will this person insure “perfectly” in the sense that the optimal value of $\alpha$ is equal to $D$? Demonstrate this rigorously, and give some intuition.

(c) Suppose that the Bernoulli utility function has the functional form $u(x) = \log(x)$. Find the optimal value of $\alpha$ for this individual.

(d) What value of $q$ will result in this individual choosing to buy exactly zero insurance claims?
Part II

Answer at most one question from Part II.
Question II.1

Consider the cost function:
\[ c(r, q) = (r_1^\alpha + r_2^\alpha)^{\frac{1}{\alpha}} q_1^{\frac{1}{2}} q_2^{\frac{1}{2}}, \]
where \( r_1 > 0 \) and \( r_2 > 0 \) are input prices, \( q_1 \geq 0 \) and \( q_2 \geq 0 \) are outputs, and \( \alpha \) is a constant parameter. This cost function is derived from a PPS that is nonempty, strictly convex, closed, and satisfies weak free disposal of output and inputs.

(a) Derive the conditional input demands given this cost function. What condition on \( \alpha \), if any, is required for these conditional input demands to exhibit valid own-price effects? Justify your answer.

(b) Assuming competitive output markets where \( p_1 > 0 \) and \( p_2 > 0 \) are the price of \( q_1 \) and \( q_2 \), find the profit-maximizing unconditional supplies assuming the solution is interior.

(c) Derive the profit-maximizing unconditional input demands (Hint: These are more quickly obtained by appealing to duality results without constructing a profit function).

(d) It is easy to verify that the conditional input demands in part (a), assuming they are correct, are homogeneous of degree zero, while the cost function is homogeneous of degree one in input prices \( r_1 \) and \( r_2 \). Show that these homogeneity properties hold in general for any conditional input demands and cost function derived from a production possibility set with \( N \) inputs and \( M \) outputs that is non-empty and closed.
Question II.2

Suppose there are $J$ firms in an industry that produce two outputs using a single input. The revenue function for the $j$th firm is

$$R^j(p_1, p_2, z_j) = \theta_j p_1^{\frac{1}{4}} p_2^\frac{3}{4} z_j^\tau,$$

where $z_j \geq 0$ is its input, $p_1 > 0$ and $p_2 > 0$ are the prices of the two outputs, and $\theta_j > 0$ is a constant parameter. This revenue function is derived from a production possibility set that is nonempty, strictly convex, closed, and satisfies weak free disposal of outputs and input.

(a) Find the revenue-maximizing distribution of inputs for the industry assuming it uses the aggregate input $z = \sum_{j=1}^J z_j > 0$ (note that the solution will be interior) and use this distribution to derive the industry’s aggregate revenue function.

(b) Derive firm $j$’s and the industry’s aggregate supply for the first output (denoted by $q_1$), conditional on the aggregate input $z$.

(c) Given the competitive price $r > 0$ for the input, find the industry’s aggregate input demand.

(d) Consider the problem in part (a) for the general revenue function $R^j(r, z^j)$ where $p \in \mathbb{R}_+^M$ and $z^j \in \mathbb{R}_+^N$. Show that the industry revenue-maximizing distribution of input $z^j(p, z)$ for $j = 1, \ldots, J$ is homogeneous of degree zero in output prices $p$. 


Part III

Answer at most one question from Part III.
Question III.1

Consider the two-player extensive form game in Figure 1.

(a) What is each player’s strategy set?

(b) Construct the normal-form game for this figure.

(c) Find all of the pure-strategy Nash equilibria in this normal-form game. Which of these equilibria are subgame perfect? Justify your answer.

(d) Find the mixed-strategy Nash equilibrium for the subgame starting after Player 1 chooses $IN$. Assuming this mixed-strategy Nash equilibrium is played instead of a pure-strategy Nash equilibrium, will Player 1 start the game by choosing $OUT$ or $IN$? Justify your answer.

Figure 1: (Player 1’s Payoff, Player 2’s Payoff)
Question III.2

Suppose there are $N > 1$ firms that compete for unskilled labor in the small town of Ames, Iowa. The marginal revenue per labor hour for the $i$th firm is a constant $r_i > 0$. The hourly wage rate for this unskilled labor is $w = h(L)$ for $L = \sum_{i=1}^{N} L_i$, where $L_i$ is the $i$th firm’s labor demand. Therefore, the $i$th firm’s profit can be written as $\pi_i = r_i L_i - h(L)L_i$.

(a) Assuming $h(L) = \alpha L$ for $\alpha > 0$:

i) Derive the total Nash equilibrium labor demand and the $i$th firm’s Nash equilibrium labor demand and profit assuming all firms choose labor simultaneously. (You may confine your attention to an interior solution).

ii) Assuming identical marginal revenues (i.e., $r_i = r$ for all $i$), how does having a large number of firms competing for labor hours affect the total Nash equilibrium labor demand, and the $i$th firm’s Nash equilibrium labor demand and profit? What is the economic interpretation of these results?

(b) Now suppose that $N = 2$ and $w = h(L)$, where $h'(L) > 0$ and $h''(L) > 0$, and the marginal revenue of labor again differs across firms (i.e., $r_1 \neq r_2$). How does an increase in Firm 1’s marginal revenue affect the Nash equilibrium labor demand for both Firm 1 and 2? Justify your answer.
Part IV

Answer at most one question from Part IV.
Question IV.1

Consider an exchange economy consisting of two consumers and two goods, $x$ and $y$. The aggregate endowment is $\omega = (6, 12)$ and the consumers’ utility functions are $U_1(x_1, y_1) = 2x_1 + y_1$ and $U_2(x_2, y_2) = x_2y_2$ respectively.

(a) Derive the contract curve for this economy. Construct a carefully labeled Edgeworth-box diagram to explain your answer. Make sure it includes at least one indifference curve for each consumer.

(b) Now suppose the individual endowments are $\omega_1 = (6, 6)$ and $\omega_2 = (0, 6)$. Derive the offer curves for the two consumers. Find the Walrasian equilibrium allocation and prices.

(c) Using the offer curves (demand functions) from part (b), and letting $z(p)$ denote aggregate excess demand, show that Walras’s law holds for this economy: for any $p \in \mathbb{R}^n_+$, $p \cdot z(p) = 0$. 
Question IV.2

Three consumers are to be asked to decide whether to provide a pre-designed public good that costs \( c = $300 \). If the public good is provided, each will be required to pay an equal share of $100. True valuations of the public good (before paying the $100) are \( r_1 = 80 \), \( r_2 = 120 \), and \( r_3 = 150 \). Each consumer knows the others’ valuations.

(a) Which outcome is Pareto optimal: provision or nonprovision of the public good?

(b) Which outcome will be selected by majority vote? Here, the two alternatives under consideration are provision (which means each voter must pay $100 to fund their share of the good) and nonprovision. Assume that everyone votes their true preferences.

(c) Now suppose each consumer announces a bid, her willingness to pay, \( b_j \) (not necessarily equal to \( r_j \)), for the public good. If \( \sum_j b_j \geq 300 \) the good is provided and each consumer pays \( b_j \). This game has many Nash equilibria, some of which lead to provision and some of which do not. Find a Nash equilibrium of each kind: one that leads to provision and one that leads to nonprovision. Tell whether the two equilibria are Pareto optimal.

(d) Now let \( v_j \) represent \( j \)'s net valuation of the public good, after paying the cost share of $100. (Thus, we have \( v_1 = -20 \), \( v_2 = 20 \), and \( v_3 = 50 \).) Write down the simplified VCG mechanism for this problem. This will consist of two mathematical objects: the decision rule as it depends on submitted sealed bids \( w_j \), not necessarily equal to the \( v_j \); and a VCG tax rule. Show that truth telling is a dominant strategy of this mechanism. Compute the outcome, including: (1) whether the public good is provided; (2) the net value received by each consumer; (3) which of the consumers if any is pivotal; and (4) the VCG tax paid by each consumer.