Ph.D. Preliminary Examination
MICROECONOMIC THEORY
Applied Economics Graduate Program
June 2015

The time limit for this exam is four hours. The exam has four sections. Each section includes two questions. Answer one question from each section. Before turning in your exam, number each page of your answers in sequential order, and identify the question (for example, III.2 for section III and question 2). Indicate, by circling below, the questions you completed. If you answer more than one question in a section and do not circle a question corresponding to that section, the first of the two that appears in your solution paper will be graded.

******************************************************************************

STUDENT ID LETTER: _______ (Fill in your code letter)

Answer one question from each section. Indicate the one you answered by circling:

Section I: Question 1 Question 2
Section II: Question 1 Question 2
Section III: Question 1 Question 2
Section IV: Question 1 Question 2

*** TURN IN THIS SHEET WITH YOUR ANSWER PAGES ***
Part I

Answer at most one question from Part I.
(Indirect utility and demand analysis for miles driven and one other good.) You are the proud owner of a very expensive Mercedes-Benz. It cost you a lot of money to buy that car, so now you spend all your money on only two things that give you utility: driving your car around, at a cost of \( p_m \) per mile driven, and buying rice to eat, which has a price of \( p_r \). Your total wealth for these two activities is denoted by \( w \). Having taken Ph.D.-level microeconomic theory you know that your indirect utility function, denoted by \( v \), is:

\[
v = -(1/\lambda)e^{-\lambda(w/p_r)} - (1/\beta)e^{\alpha+\beta(p_m/p_r)},
\]

where \( \lambda, \alpha \) and \( \beta \) are parameters (constants) in the indirect utility function.

(a) The indirect utility function should be homogeneous of some degree \( x \) in prices and wealth. What is \( x \)? Does this indirect utility function satisfy the homogeneity restriction? Do you need to impose any constraints on \( \lambda, \alpha \), and/or \( \beta \) to ensure that this indirect utility function satisfies the homogeneity restriction?

(b) Another property of indirect utility functions concerns whether they increase or decrease when prices and wealth change. What exactly are these properties? Does the above indirect utility function satisfy them? Do you need to impose any constraints on \( \lambda, \alpha \) and/or \( \beta \) to ensure that this indirect utility function satisfies these properties? You can assume that \( w > 0, p_m > 0, p_r > 0 \), and that \( w \geq p_m \) and \( w \geq p_r \) (you have enough wealth to drive at least one mile or buy at least one unit of rice). Note that you do not need the complete answer for b) to do the rest of this problem, so if you have trouble you can move on to part (c).

(c) Use a property of indirect utility functions that allows you to derive your Walrasian demand for miles driven in your new car. What is the property that you used? What is your demand for miles on your new car? Denote the demand for miles as \( x_m \).

(d) Finally, use your answer for (c) to derive the own-price elasticity of the demand for miles \( (x_m) \) and the wealth elasticity of the demand for miles \( (x_m) \). Use your answer for the wealth elasticity to assess whether the preference relation underlying this demand system is homothetic.
Question I.2

(Risk and Expected Utility.) Consider an individual who faces a risky environment and has the following Bernoulli utility function (where $x$ is an amount of money):

$$u(x) = x^\alpha, \quad \alpha > 0, \quad x \geq 0.$$ 

(a) Suppose that this individual faces the following “lottery” for possible values of $x$, where $x$ takes only 3 values, 0, 1 or 2:

- $\text{Prob}[x = 0] = 1/4$
- $\text{Prob}[x = 1] = 1/2$
- $\text{Prob}[x = 2] = 1/4$.

What is this person’s expected utility from this lottery?

(b) What amount of money is the “certainty equivalent” for this person for this lottery? Your answer should be a function of $\alpha$.

(c) This question refers to the Bernoulli utility function above but has nothing to do with the lottery that was referred to in parts (a) and (b). For the above Bernoulli utility function, what is the “probability premium” for a fixed amount of money ($x$) and some positive number ($\varepsilon$)? Your answer should be a function of $x$ and $\varepsilon$.

(d) Is it possible that the probability premium for this Bernoulli utility function is negative? There are two ways to answer this question. First, you could appeal to some “equivalence properties” of Bernoulli utility functions, in which case you do not need to make use of your answer (or lack of an answer) to part (c). Alternatively, you could use your answer to part (c). Either way is fine, and you do not get any extra points for doing both (so just do one).

(e) Finally, consider another “lottery” for this individual. Let $x$ be a random number drawn between 5 and 10. That is, $x$ follows a uniform distribution between 5 and 10, such that

$$f(x) = \begin{cases} 
0.2 & \text{if } 5 \leq x \leq 10 \\
0 & \text{if } x < 5 \text{ or } x > 10 .
\end{cases}$$

For the same Bernoulli utility function, what is the “certainty equivalent” of this lottery? Again, your answer should be a function of $\alpha$. 
Part II

Answer at most one question from Part II.
Question II.1

Consider the production possibility set (PPS):

$$PPS = \{(q, -z) \in \mathbb{R}_+ \times \mathbb{R}^2_\alpha : (z_1^{\alpha_1} + z_2^{\alpha_2})^{\frac{1}{\rho}} \geq q^{\frac{3}{4}}_1 + q^{\frac{1}{4}}_2\},$$

where $q_1$ and $q_2$ are outputs; $z_1$ and $z_2$ are inputs; and $\alpha_1 > 0$ and $\alpha_2 > 0$ are constant parameters. This PPS is nonempty, strictly convex, closed, and satisfies weak free disposal in inputs and outputs.

(a) Derive the input distance function assuming $q_1 > 0$ and $q_2 > 0$ and use it to calculate the input elasticity of scale.

(b) Assuming $\alpha_1 = \alpha_2$, the input distance function for this PPS will yield the cost function

$$c(r, q) = (r_1^{\alpha} + r_2^{\alpha})^\frac{1}{\alpha} \left(q^{\frac{3}{4}}_1 + q^{\frac{1}{4}}_2\right),$$

where $r_1 > 0$ and $r_2 > 0$ are the price of $z_1$ and $z_2$, and $\alpha = \rho/(\rho - 1)$. Derive the conditional input demands given this cost function.

(c) Assuming competitive output markets where $p_1 > 0$ and $p_2 > 0$ are the price of $q_1$ and $q_2$, find the profit-maximizing supplies and unconditional input demands assuming the solution is interior. (Hint: the unconditional profit-maximizing input demands are more quickly obtained by appealing to duality results without constructing a profit function.)

(d) Show that the general cost function $c(r, q)$ with input prices $r \in R^{N}_+$ and outputs $q \in R^{M}_+$ is concave in prices $r$ given the PPS is nonempty, strictly convex, closed, and satisfies weak free disposal in output and inputs.
Question II.2

Suppose there are \( J \) firms in an industry that produce identical output using the same two inputs. The cost of production for the \( j \)th producer is

\[
c^j(r, q_j) = \frac{(r_1^{\alpha_j} + r_2^{\alpha_j})^{\frac{1}{\alpha_j}}}{2} q_j^2,
\]

where \( q_j \geq 0 \) is its output, \( r_1 > 0 \) and \( r_2 > 0 \) are the prices of the two inputs, and \( \alpha_j > 0 \) is a constant parameter. This cost function is derived from a production possibility set that is nonempty, strictly convex, closed, and satisfies weak free disposal in output and inputs.

(a) Find the cost-minimizing distribution of output for the industry assuming it produces aggregate output \( q = \sum_{j=1}^{J} q_j > 0 \) (note that the solution will be interior) and use this distribution to derive the industry’s aggregate cost function.

(b) Derive producer \( j \)'s and the industry’s aggregate unconditional demand for the first input (denoted by \( z_1 \)) given aggregate output \( q \).

(c) Given the competitive price \( p > 0 \) for output, find the industry’s aggregate supply.

(d) Consider the problem in part (a) for the general cost function \( c^j(r, q^j) \) where \( r \in \mathbb{R}_+^N \) and \( q^j \in \mathbb{R}_+^M \). Show that the industry cost-minimizing distribution of output \( q^j(r, q) \) for \( j = 1 \ldots, J \) is homogeneous of degree zero in input prices \( r \).
Part III

Answer at most one question from Part III.
Question III.1

Consider the two-player extensive form game in Figure 1.

(a) What is each player’s strategy set?

(b) Construct the normal form game for this figure.

(c) Assuming $x = 5$, find all of the pure-strategy Nash equilibria in this normal form game. Which of these equilibria are subgame perfect? Justify your answer.

(d) Assuming $x > 0$, find the mixed-strategy Nash equilibrium for the subgame starting after both players choose $P$. For what $x$ (if any), will both players choosing $P$ be part of a subgame perfect equilibrium?

Figure 1: (Player 1’s Payoff, Player 2’s Payoff)
Question III.2

Suppose there are \( N > 1 \) firms seeking to secure a lucrative government contract worth \( V > 0 \). The \( n \)th firm invests effort \( x_n \geq 0 \) to influence the probability it wins the contract, such that the probability firm \( n \) wins is \( p_n = x_n / (\sum_{i=1}^{N} x_i) \). The marginal cost of effort is a constant \( c_n > 0 \), so the \( n \)th firm’s expected payoff can be written as

\[
\pi_n = \frac{x_n}{\sum_{i=1}^{N} x_i} V - c_n x_n.
\]

(a) Derive the total Nash equilibrium effort, and the \( n \)th firm’s Nash equilibrium effort and expected payoff, assuming effort is chosen simultaneously and the solution is interior.

(b) Assuming identical marginal costs (i.e., \( c_n = c \) for all \( n \)), how does having a large number of firms competing for the contract affect the total Nash equilibrium effort, and the \( n \)th firm’s Nash equilibrium effort and expected payoff? What is the economic interpretation for these results?

(c) Now suppose that \( N = 2 \), the marginal cost of effort can again differ, and that the probability of \( n \) winning is generally \( p_n(x_n, x_{\sim n}) \), where \( x_{\sim n} \) is the effort of firm \( n \)’s opponent, and where \( \partial p_n(x_n, x_{\sim n})/\partial x_n > 0 \) and \( \partial^2 p_n(x_n, x_{\sim n})/\partial x_n^2 < 0 \). Firm \( n \)’s payoff can now be written as

\[
\pi_n = p_n(x_n, x_{\sim n}) V - c_n x_n.
\]

Assuming an interior solution, derive the condition required to guarantee that an increase in \( V \) increases firm 1’s equilibrium effort. Is this condition more likely to be satisfied when firm 1’s marginal cost is higher or lower than firm 2’s? Explain.
Part IV

Answer at most one question from Part IV.
Question IV.1

Consider the following $2 \times 2$ competitive exchange economy, with consumers $j = 1, 2$ and goods $x$ and $y$. The consumers’ preferences are given by

$$U_1(x_1, y_1) = x_1 y_1 \quad \text{and} \quad U_2(x_2, y_2) = \min\{2x_2, y_2\}.$$ 

Endowments are $\omega_1 = (4, 0)$ and $\omega_2 = (0, 4)$.

(a) Find the set of strongly Pareto-optimal allocations. Is this set equal to the set of weakly Pareto-optimal allocations? Construct an Edgeworth-box diagram depicting the economy.

(b) Derive the two consumers’ offer curves. Find a Walrasian equilibrium for the economy. True or False: The equilibrium is unique. (Find more than one equilibrium if false; prove uniqueness if true.)

(c) Derive the set of allocations that can be supported as Walrasian equilibrium allocations, at strictly positive prices $p \gg 0$, with a suitable redistribution of the initial endowment.

(d) Find the allocation that would be selected by a social planner whose objective is to maximize the social-welfare function $W = \min\{U_1, U_2\}$. 
Question IV.2

Recall Arrow’s theorem: Suppose that preferences \( P_j \) are transitive, and that \( |X| \geq 3 \). Then any social welfare function \( f(\{P_j\}_{j=1}^n) \) satisfying (I), (U), and (P) is dictatorial.

Answer either Part (a) or part (b). The latter refers to the proof of Arrow’s theorem given in J. Geanakoplos’s 2005 paper.

**Part (a)**

(a.1) Show that there is some pair \( a, b \in X \) such that some person \( j \in J \) is semidecisive (SD) over \( a \) and \( b \).

(a.2) Consider a social-choice setting with three individuals and two alternatives, \( X = \{a, b\} \) and in which each individual’s ordering is strict. Prove that majority rule is not dictatorial for this problem.

**Part (b)**

(b.1) Prove that there is an “extremely pivotal” voter, who can, by unilaterally changing her vote, move some alternative from the bottom of the social ranking to the top. For this proof, begin with a profile in which every voter places some alternative \( c \) last. Move \( c \) from last to first in the ranking of one voter, then another, and so on. There must be some voter \( j^* \) such that raising \( c \) to the top of \( j^* \)’s ranking caused \( c \) to move from last to something higher than last in the social ranking. Show that when \( c \) moves to the top of \( j^* \)’s ranking, it must move to the top of the social ranking.

(b.2) Show that \( j^* \) from part (b.1) is decisive over all pairs \( a \) and \( b \) not involving \( c \).