Ph.D. Preliminary Examination
MICROECONOMIC THEORY
Applied Economics Graduate Program
August 2017

The time limit for this exam is four hours. The exam has four sections. Each section includes two questions. Answer one question from each section. Before turning in your exam, number each page of your answers in sequential order, and identify the question (for example, III.2 for section III and question 2). Indicate, by circling below, the questions you completed. If you answer more than one question in a section and do not circle a question corresponding to that section, the first of the two that appears in your solution paper will be graded.

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STUDENT ID LETTER: _______ (Fill in your code letter)

Answer one question from each section. Indicate the one you answered by circling:

Section I:   Question 1   Question 2
Section II:  Question 1   Question 2
Section III: Question 1   Question 2
Section IV:  Question 1   Question 2

*** TURN IN THIS SHEET WITH YOUR ANSWER PAGES ***
Part I

Answer at most one question from Part I.
Question I.1

Consider a consumer with the following utility function for three goods, $x_1$, $x_2$ and $x_3$:

$$u = \beta_1 \ln(x_1) + \beta_2 \ln(x_2) + \beta_3 x_3,$$

where $\ln(\ )$ is the natural logarithm function.

(a) Derive the Walrasian demands functions for $x_1$, $x_2$ and $x_3$. Assume that both wealth $(w)$ and all prices $(p_1, p_2$ and $p_3)$ are strictly greater than 0. For this part, you can assume an interior solution (strictly positive demand for all three goods).

(b) Are the Walrasian demands that you derived for part (a) necessarily positive when all prices and wealth are strictly greater than 0? The answer to this question should be very brief. Also, the Walrasian demand functions you just derived correspond to a particular type of preferences. What is this type of preferences? Just state the answer, there is no need to demonstrate it.

Note that parts (c), (d) and (e) of this question do not depend on your answer to part (b), so if you are not sure about part (b) you can still go on to the remaining parts of this question.

(c) Use your answer to part (a) to derive the indirect utility function for this consumer.

(d) Use your answer to part (c) to derive the expenditure function that corresponds to the original utility function.

(e) Finally, use your expenditure function to derive the Hicksian demands for $x_1$, $x_2$ and $x_3$, which can be denoted as $h_1(p_1, p_2, p_3, u)$, $h_2(p_1, p_2, p_3, u)$ and $h_3(p_1, p_2, p_3, u)$, respectively.
Question I.2

You work in a bank, and you stole $30,000 from the bank. The bank examiner will come tomorrow and your theft of $30,000 will be detected unless you pay back the money tonight. But you have already spent $20,000, so now you have only $10,000 and you have no other source of money. This means that the only way that you can pay back the $30,000 is to make a gamble that will have a payoff of $30,000. If you do not pay back all $30,000 you will be put in prison with a consumption level \( x \) of 0.

(a) Consider first the types of gambles that are available to you. You are fortunate that there are “fair” gambles available, so that the expected value of any gamble you choose is equal to the amount gambled. Assume that all possible gambles take the simple form that they pay you $30,000 if you win, which happens with probability \( \pi \), and they pay $0 if you lose, which happens with probability \( 1 - \pi \). The probability \( \pi \) will vary depending on the amount gambled. Let \( G \) denote the amount of money that you gamble. Using the assumption that any gamble you make for any value of \( G \) is a fair gamble, express \( \pi \) as a function of \( G \) and one or more fixed numbers.

(b) Your friend knows that you are very risk averse, and that friend advises you to put all $10,000 into the gamble, that is to set \( G = 10,000 \), because this will minimize the probability that you go to jail. Is your friend correct that this will minimize the probability of going to jail? Refer to your answer for part (a). Also, given that you are risk averse, is this a good strategy? This second question can be answered in words only; no need for any math.

(c) Given your answer to part (a), what is your expected utility when you have a Bernoulli utility function of \( v(x) \), where \( x \) is consumption, and you choose to undertake a gamble with an amount of money \( G \), where \( 0 \leq G \leq 10,000 \)? Denote expected utility by \( EU \). Note that your answer should be function of \( G \), but not of \( \pi \). Do not try to solve for expected utility yet; that will be done in parts (d) and (e).

(d) Suppose that you have a Bernoulli utility function \( v(x) = x^{0.5} \). Are you risk averse, risk loving, or risk neutral? Using your answer to (c), what is your optimal gamble \( (G) \) for maximizing your expected utility with this Bernoulli utility function?

(e) Suppose that you have a Bernoulli utility function \( v(x) = x \). Are you risk averse, risk loving, or risk neutral? Using your answer to (c), what is your optimal gamble \( (G) \) for maximizing your expected utility with this Bernoulli utility function?

(f) Consider your answers to (d) and (e). Usually, we would expect to see that a higher level of risk aversion means less willingness to make a large gamble. Is that what you find in this case? Explain the intuition in what you find.
Part II

Answer at most one question from Part II.
Question II.1

Consider the production possibility set (PPS) for producer $j$:

$$
\text{PPS} = \{(q, -z) \in \mathbb{R}_+ \times \mathbb{R}^2 : \sqrt{z_1 z_2} \geq q^{\lambda_j}\},
$$

where $z_1$ and $z_2$ are inputs; $q$ is output; and $\lambda_j > 1$ is a constant parameter. This PPS is nonempty, strictly convex, and closed.

(a) Show that this PPS satisfies weak free disposal of inputs and outputs.

(b) Derive the input distance function for this PPS assuming $q > 0$ and use this input distance function to calculate the marginal rate of transformation between $z_1$ and $q$.

(c) The profit function for this PPS is

$$
\pi^j(p, r_1, r_2) = \frac{p^{\gamma_j}}{\gamma_j} \left(\frac{1}{2 \lambda_j \sqrt{r_1 r_2}}\right)^{\gamma_j - 1},
$$

where $\gamma_j = \frac{\lambda_j}{\lambda_j - 1}$, $r_1 > 0$ and $r_2 > 0$ are competitive input prices, and $p > 0$ is the competitive output price. Use this profit function to find the producer’s unconditional input demands.

(d) Suppose there are $J$ producers in the industry. What is the aggregate industry supply and profit?

(e) If you are correct for part (d), it should be easy to see that the aggregate industry supply is homogeneous of degree zero in input and output prices, while the aggregate industry profit is homogeneous of degree one in input and output prices. Show that this result holds in general if we have $J$ producers in an industry with $N$ inputs and $M$ outputs. You may assume that each producer in the industry has a production possibility set that is nonempty, convex, closed, and satisfies weak free disposal of outputs and inputs as well as all these assumptions imply in terms of a producer’s unconditional supplies and profit: $q^j(p, r)$ and $\pi^j(p, r)$ for $p \in \mathbb{R}^M_+, r \in \mathbb{R}^N_+$ and $j = 1, \ldots, J$. 

Question II.2

Consider an uncertain world with only two mutually exclusive states denoted by \( \text{a} \) and \( \text{b} \). A producer produces output \( q_a \geq 0 \) in state \( \text{a} \) and \( q_b \geq 0 \) in state \( \text{b} \) with two inputs, \( z_1 \) and \( z_2 \), at a cost of

\[
C(r_1, r_2, q_a, q_b) = 2r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}(q_a^2 + q_b^2),
\]

where \( r_1 > 0 \) and \( r_2 > 0 \) are the prices of inputs \( z_1 \) and \( z_2 \). This cost function is derived from a production possibility set that is nonempty, strictly convex, closed, and satisfies weak free disposal in outputs and inputs. Given this cost function, the competitive producer’s profits in states \( \text{a} \) and \( \text{b} \) are

\[
\pi_a = p_a q_a - 2r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}(q_a^2 + q_b^2), \quad \text{and} \quad \pi_b = p_b q_b - 2r_1^{\frac{1}{2}}r_2^{\frac{1}{2}}(q_a^2 + q_b^2),
\]

where \( p_a > 0 \) and \( p_b > 0 \) are the price of output in states \( \text{a} \) and \( \text{b} \).

(a) For the state contingent output \( q_a \) and \( q_b \), what are the producer’s conditional input demands?

(b) Suppose a risk averse producer’s utility of profit function is

\[
W(\pi_a, \pi_b) = 3 - e^{-\mu \pi_a} - 2e^{-\mu \pi_b},
\]

where \( \mu > 0 \).

(i) What is the producer’s subjective probabilities for state \( \text{a} \) and \( \text{b} \)?

(ii) For \( \pi_a \) and \( \pi_b \) in general, what is the producer’s certainty equivalent profit? Does the certainty equivalent profit increase or decrease as \( \mu \) increases? What does this result imply about the relationship between a producer’s risk aversion and \( \mu \)?

(c) Suppose instead that the producer is risk neutral with subjective probability beliefs \( \phi_a > 0 \) and \( \phi_b > 0 \) for state \( \text{a} \) and \( \text{b} \).

(i) Find this producer’s optimal outputs for states \( \text{a} \) and \( \text{b} \). You can assume the solution is interior.

(ii) Given these optimal outputs, what are the producer’s unconditional input demands?
Part III

Answer at most one question from Part III.
Question III.1

Simplex Solutions ("the firm") is the sole supplier of simulation software. The cost of developing simulation software is \( C \). Once developed, the firm can produce additional units of simulation software for no cost. The firm has two potential customers, Mr. \( X \) and Ms. \( Y \), and each has demand for a single unit of simulation software. The value of the simulation software for Mr. \( X \) is \( V_x \) and for Ms. \( Y \) is \( V_y \), with \( V_x > V_y \).

(a) Suppose the firm knows both \( V_x \) and \( V_y \). Suppose the firm sets a price \( P \) and that each potential customer then chooses whether or not to purchase simulation software. Assuming it is profitable to develop the simulation software, find all potential equilibrium prices and the range of parameter values for \( V_x \) and \( V_y \) that support any given equilibrium price. Under what set of parameter values for \( C \), \( V_x \) and \( V_y \) would the firm find it profitable to develop the simulation software?

(b) Now suppose that the firm can sell simulation software in period 1 and in period 2. Let \( P_1 \) represent the first-period price and let \( P_2 \) represent the second-period price. Suppose that Mr. \( X \) has a pressing need to get the simulation software as soon as possible. Mr. \( X \) has a value of \( V_x \) if he purchases software in period 1, but he only values the simulation software at \( \gamma V_x \), \( 0 < \gamma < 1 \), if he purchases software in period 2. Ms. \( Y \) is indifferent about when she purchases the software and gains value \( V_y \) with purchase of software either in period 1 or period 2. Assume the firm knows \( V_x \), \( V_y \) and \( \gamma \). Suppose that the firm discounts sales in period 2 by \( \delta \), with \( \gamma < \delta < 1 \).

(i) Find a set of prices \((P_1, P_2)\) that induce both Mr. \( X \) and Ms. \( Y \) to purchase software in period 1 and maximizes the present value of profits for the firm given the constraint that it sells to both customers in period 1. What is the present value of profit for the firm when it sets these prices?

(ii) Find a set of prices \((P_1, P_2)\) that would induce Mr. \( X \) to purchase software in period 1 and Ms. \( Y \) to purchase software in period 2 and maximizes the present value of profits for the firm, given the constraint that it sell to Mr. \( X \) in period 1 and Ms. \( Y \) in period 2. What is the present value of profit for the firm when it sets these prices?

(iii) Under what values of \( V_x \), \( V_y \), \( \gamma \), and \( \delta \) is it most profitable to sell software to Mr. \( X \) in period 1 and to Ms. \( Y \) in period 2 rather than sell to both customers in period 1? Explain the economic intuition for your answer.

(c) Finally, suppose Mr. \( X \) has changed jobs and no longer wishes to buy simulation software so that the only potential customer is Ms. \( Y \). Suppose that nature draws \( V_y \) from a uniform distribution on \([0, 1]\) and that Ms. \( Y \) knows the value of \( V_y \) but that the firm does not. Suppose that nature draws \( C \) from a uniform distribution on \([0, 1]\) and assume that the firm knows the value of \( C \) but that Ms. \( Y \) does not. Ms. \( Y \) chooses a price at which she is willing to buy, \( P_y \), and the firm choose a price at which it is willing to sell, \( P_s \). Prices are chosen simultaneously. If \( P_y \geq P_s \) then trade occurs at price \( P = 0.5(P_y + P_s) \). If \( P_y < P_s \) then no trade occurs. Find a Bayesian Nash equilibrium to this game when the players play linear strategies as a function of values: \( P_y = a + bV_y \) and \( P_s = g + hC \).
Question III.2

Das Boot (DB) and Intelligent Design (ID) each design software for use in personal computers. After investing a large fixed cost to make the software, the marginal cost of selling an additional unit of software is zero. DB has the following demand curve:

\[ X(P_x, P_y) = a - \frac{1}{2}P_x + bP_y, \]

where \( X \) is demand for DB software, \( P_x \) is the price of DB software, and \( P_y \) is the price of ID software, with \( a > 0 \) and \( b > 0 \). ID has a demand curve given by:

\[ Y(P_x, P_y) = \alpha - \frac{1}{2}P_y + \beta P_x, \quad \alpha > 0, \beta > 0, \]

where \( Y \) is the demand for ID software. Assume that \( \beta b < \frac{1}{2} \).

(a) Solve for the Bertrand (Nash) equilibrium when firms choose price simultaneously.

(b) Now suppose that DB sets price prior to ID, i.e., DB is the leader and ID is the follower. Solve for subgame perfect equilibrium prices. Are equilibrium prices higher, lower, or the same in part (b) as compared to part (a)? Provide economic intuition for this result.

(c) Now suppose that DB and ID set prices simultaneously but there is asymmetric information about demand. Suppose that \( a = 1 \) with probability \( 1/3 \), and \( a = 2 \) with probability \( 2/3 \). Assume that DB knows whether \( a = 1 \) or \( a = 2 \) but that ID does not. Assume that \( \alpha = 1, b = \beta = 0.5 \), and that both firms know this. Solve for Bayesian Nash equilibrium.

(d) Now suppose that DB and ID compete over two periods. Software becomes more attractive the more others use it (“network externalities”). Period 1 demand for DB and ID is given by:

\[ X_1(P_{X_1}, P_{Y_1}) = 1 - \frac{P_{X_1}}{2} + \frac{P_{Y_1}}{2} \quad \text{and} \quad Y_1(P_{X_1}, P_{Y_1}) = 1 - \frac{P_{Y_1}}{2} + \frac{P_{X_1}}{2}. \]

Period 2 demand for DB and ID is:

\[ X_2(P_{X_2}, P_{Y_2} \mid P_{X_1}, P_{Y_1}) = 1 - \frac{P_{X_2}}{2} + \frac{P_{Y_2}}{2} + \frac{X_1}{2} \]

\[ = 1 - \frac{P_{X_2}}{2} + \frac{P_{Y_2}}{2} + \frac{1}{2} - \frac{P_{X_1}}{4} + \frac{P_{Y_1}}{4} \quad \text{and} \]

\[ Y_2(P_{X_2}, P_{Y_2} \mid P_{X_1}, P_{Y_1}) = 1 - \frac{P_{Y_2}}{2} + \frac{P_{X_2}}{2} + \frac{Y_1}{2} \]

\[ = 1 - \frac{P_{Y_2}}{2} + \frac{P_{X_2}}{2} + \frac{1}{2} - \frac{P_{Y_1}}{4} + \frac{P_{X_1}}{4}. \]

For simplicity, assume there is no discounting between period 1 and period 2. Solve for the subgame perfect equilibrium prices \( P_{X_1}, P_{Y_1}, P_{X_2}, \) and \( P_{Y_2} \).
Part IV

Answer at most one question from Part IV.
Question IV.1

Consider the following $2 \times 2$ competitive exchange economy, with consumers $j = 1, 2$ and goods $x$ and $y$. The consumers’ preferences are given by

$$U_1(x_1, y_1) = x_1^{1/3} y_1^{2/3} \quad \text{and} \quad U_2(x_2, y_2) = x_2^{2/3} y_2^{1/3}.$$ 

Consumers’ endowments for the economy are $\omega_1 = (10, 0)$ and $\omega_2 = (0, 5)$.

(a) Construct a carefully labeled Edgeworth-box diagram depicting the economy, including the endowment and at least one indifference curve for each consumer.

(b) Find the competitive-equilibrium price vector and allocation for the economy. Add the equilibrium allocation, and the equilibrium budget line through $\omega$, to your diagram.

(c) Suppose a social planner wishes to allocate the aggregate endowment of $(10, 5)$ (we no longer specify how the endowment is divided among the consumers) to maximize consumer 1’s utility while holding 2’s utility at

$$U_2 = \left( \frac{2000}{27} \right)^{1/3}.$$ 

Find the assignment of goods to consumers that solves the planner’s problem. Show that the solution is Pareto optimal and that it agrees with the equilibrium allocation you found in part (b) of this question.
Question IV.2

Consider the usual social-choice setting with a finite set of $m$ alternatives $X$ and a finite set of individuals $J = \{1, \ldots, n\}$. Preferences $P_j$ are strict on $X$ and we say that a profile is $\{P_j\}_{j=1}^n$. Answer part (a) and either (b) or (c).

(a) State Arrow’s theorem. Include definitions of each of the four axioms as well as the social welfare function at issue.

(b) Provide a proof of step 2 of the first proof of Arrow’s Theorem. That is, take as given a person $j$ who, from step 1, is semi-decisive (SD) over some pair $a, b$. Show that this person $j$ is decisive over all $x, y \in X$ (that is, $j$ is a dictator).

(c) Take as given a person $j^*$ who, from steps 2 and 3, is extremely pivotal (that is, by unilaterally changing his vote, $j$ can move some alternative from the bottom of the social ranking to the top). Provide a proof of steps 3 and 4 of the second proof of Arrow’s Theorem. Recall that step 3 establishes that the extremely pivotal voter is decisive on all pairs not involving alternative $c$, which was moved from bottom to top of each voter’s rankings in step 2. Step 4 establishes that this voter is also decisive on any pair involving $c$. 