Ph.D. Preliminary Examination
MICROECONOMIC THEORY
Applied Economics Graduate Program
June 2017

The time limit for this exam is four hours. The exam has four sections. Each section includes two questions. Answer one question from each section. Before turning in your exam, number each page of your answers in sequential order, and identify the question (for example, III.2 for section III and question 2). Indicate, by circling below, the questions you completed. If you answer more than one question in a section and do not circle a question corresponding to that section, the first of the two that appears in your solution paper will be graded.

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STUDENT ID LETTER: ______ (Fill in your code letter)

Answer one question from each section. Indicate the one you answered by circling:

Section I: Question 1 Question 2
Section II: Question 1 Question 2
Section III: Question 1 Question 2
Section IV: Question 1 Question 2

*** TURN IN THIS SHEET WITH YOUR ANSWER PAGES ***
Part I

Answer at most one question from Part I.
Question I.1

(Consumer Demand and Utility under Two Food-Assistance Programs.) Consider a utility function, which is a function of 2 goods, \( F \) (food) and \( N \) (nonfood):

\[
u = F^\beta N^{1-\beta},
\]

where \( 0 < \beta < 1 \). The consumer has wealth \( w \) to purchase these two goods, where the price of \( F \) is \( p_F \) and the price of \( N \) is \( p_N \). In this question, you will compare two different policies to increase the welfare of poor people: giving them a physical quantity of food and giving them a cash grant to buy whatever they want to buy.

(a) To see the difference between the two programs, draw a diagram with consumption of food \((F)\) on the horizontal axis and consumption of nonfood \((N)\) on the vertical axis. Assume that one program, call it the “food grant” program, provides the consumer 50 units of food. The consumer cannot sell the food from the food-grant program; he or she must consume it (and can buy more food if he or she wants). The other program, call it the “cash grant” program, provides the consumer \(50 \times p_F\) worth of cash, and the consumer can spend it on either food or nonfood. Denote this cash transfer by \(c\). Show in your diagram the budget sets for both of these programs, labeling the values (as functions of \(w, c, p_F, \) and \(p_N\)) where they touch the two axes and indicating the values on the axes corresponding to the “kink” in one of the budget sets.

(b) For the utility function given above, work out the demand for \(F\) and \(N\) for the case of the cash-grant program. These demands should be functions of \(\beta, w, c, p_F\) and \(p_N\). You can assume interior solutions for both.

(c) Using your answer immediately above, for what value of \(\beta\) will the consumer choose to consume exactly 50 units of food \((F)\)? Your answer should show this value of \(\beta\) as a function of one or more of the following: \(w, c, p_F, \) and \(p_N\).

(d) For what values of \(\beta\) will the consumer be worse off under the food-grant program relative to the cash-grant program? It may be useful to refer to your diagram in the first part of this problem. Again, your answer should show a relationship between \(\beta\) and one or more of the following: \(w, c, p_F, \) and \(p_N\).

(e) Consider a consumer who would have purchased 25 units of food under the cash-grant program. How much would his or her utility change if the cash-grant program were replaced with the food-grant program? (You do not need to simplify your answer to this question.)

(f) How much additional cash would this consumer need under the food-grant program to have the same utility that he or she had under the cash-grant program? You can denote this amount by “\(ac\)”. Show that \(ac > 0\). [Hint: Do NOT try to use the compensating variation formula given in class, since that is for a price change. Instead you need to work this out using the utility function. You can assume that even with the additional cash the consumer will not consume more than 50 units of food.]
Question I.2

(Risk Aversion in Income and in Prices.) Consider an individual with wealth $w$ who has a utility function that is a function of two goods, $x$ and $y$:

$$ u(x, y) = x + \alpha \ln(y), \quad \text{where } \alpha > 0. $$

For simplicity, assume that the price of $x$ is 1, and the price of $y$ is $p$, where $p > 0$.

(a) Solve for the demands for both $x$ and $y$ as functions of $p$ and $w$. What restriction, if any, is needed on $\alpha$ to assure that both demands are $\geq 0$?

(b) Based on your answers above, derive the indirect utility function for this consumer, which is a function of $p$ and $w$, and can be denoted as $v(p, w)$.

(c) Assume that prices are fixed. Then your answer above can be considered to be a Bernoulli utility function in $w$. In class we used $x$ to denote the amount of money in the Bernoulli utility and denoted it by $u(x)$, but here we are using somewhat different notation $v(w)$, where $v$ replaces $u$ and $w$ replaces $x$ (and prices are fixed, at least for now, and so they are treated as part of the $v(\cdot)$ functional form). Is the consumer with the Bernoulli utility function given in your answer above (strictly) risk averse, risk neutral, or (strictly) risk loving?

(d) Attitudes towards risk can also be defined with respect to prices in the indirect utility function you derived above. Holding $w$ constant, what is the second derivative of the indirect utility function with respect to $p$? Is the consumer (strictly) risk averse, risk neutral, or (strictly) risk loving with respect to $p$? To help explain your answer you can draw a simple graph of (indirect) utility with respect to $p$, assuming that $w$ is constant.

(e) Is your answer regarding the consumer’s attitude toward risk in prices ($p$) surprising given your answer to the consumer’s attitude toward risk in wealth ($w$)? In either case, give an intuitive explanation for the consumer’s attitude towards risk in $p$ by comparing the following two scenarios. In Scenario 1, the price of $y$ has the value $p$ with 100% certainty. In Scenario 2, the price of $y$ takes one of either two values, $p \times (1/2)$ and $p \times (3/2)$, each with 50% probability.
Part II

Answer at most one question from Part II.
Question II.1

Consider the production possibility set (PPS):

$$\text{PPS} = \left\{ (q, -z) \in \mathbb{R}_+ \times \mathbb{R}_+^2 : z_1^\alpha z_2^\beta \geq q_a + \sqrt{q_a q_b} + q_b \right\},$$

where $z_1$ and $z_2$ are inputs; $q_a$ and $q_b$ are outputs; and $\alpha > 0$ and $\beta > 0$ are constant parameters. This PPS is nonempty, strictly convex, closed, and satisfies weak free disposal in outputs and inputs.

(a) Under what additional conditions on $\alpha$ and $\beta$ (if any) will this technology exhibit constant returns to scale? Justify your answer.

(b) Derive the input distance function for this PPS assuming $q_a > 0$ or $q_b > 0$. Derive the marginal rate of technical substitution between $z_1$ and $z_2$ using this input distance function.

(c) The cost function for this PPS is

$$C(r_1, r_2, q_a, q_b) = \Omega r_1^{\alpha/\alpha+\beta} r_2^{\beta/\alpha+\beta} (q_a + \sqrt{q_a q_b} + q_b)^{1/\alpha+\beta},$$

where $r_1 > 0$ and $r_2 > 0$ are input prices and $\Omega = \left( \frac{\alpha}{\beta} \right)^{\beta/\alpha+\beta} + \left( \frac{\beta}{\alpha} \right)^{\alpha/\alpha+\beta}$. Use this cost function to find the producer’s conditional input demands.

(d) Using the cost function from part (c), suppose $q_a$ represents output in state $a$ and $q_b$ represents output in state $b$ of an uncertain world with only two states. Let $p_a > 0$ and $p_b > 0$ be the price of output in these two states, and $R_a \geq 0$ and $R_b \geq 0$ be the revenue in these two states.

(i) Set up the optimization problem for the revenue cost function and derive it.  
**HINT:** Think before you act. This problem is simpler than it might first appear.

(ii) Derive the certainty equivalent revenue given this revenue cost function.
Question II.2

Consider a world with only two states denoted by $g$ for good and $b$ for bad. A producer’s profits in the good and bad state are

$$\pi_g = \mu(z) + \frac{h(z)}{\phi_g} - rz \quad \text{and} \quad \pi_b = \mu(z) - \frac{h(z)}{\phi_b} - rz,$$

where $z \in \mathbb{R}_+$ is an input, $\mu(z) > 0$ is certain revenue given $z$; $\frac{h(z)}{\phi_g} > 0$ and $-\frac{h(z)}{\phi_b} < 0$ are revenue shocks in the good and bad states of the world given $z$; and $r > 0$ is the competitive price of the input. Assume $\mu(z)$ and $h(z)$ are continuous and twice differentiable for all $z$.

(a) Suppose $W(\pi_g, \pi_b) = \alpha \pi_g + \sqrt{\pi_g \pi_b} + \beta \pi_b$ is the risk-averse producer’s utility of profit.

(i) What are its subjective probabilities for the good and bad states?

(ii) For $\pi_g$ and $\pi_b$ in general, what is the producer’s absolute risk premium given these preferences?

(b) Suppose instead that there are two types of producers. The first maximizes its expected profit where $\phi_g > \phi_b > 0$ are its subjective probabilities such that $\phi_g + \phi_b = 1$. The second maximizes the general utility of profit function $W(\pi_g, \pi_b)$ where $\frac{\partial W(\pi_g, \pi_b)}{\partial \pi_g} > 0$ and $\frac{\partial W(\pi_g, \pi_b)}{\partial \pi_b} > 0$. Furthermore, assume this producer is risk averse such that this utility function is generalized Schur-concave with respect to the probabilities $\phi_g$ and $\phi_b$.

(i) Set up the profit-maximizing producer’s optimization problem and derive the first-order condition for the interior solution.

(ii) Set up the utility-maximizing producer’s optimization problem and derive the first-order condition for the interior solution.

(iii) Assuming $\mu'(z) > 0$ and $\mu''(z) < 0$, what conditions on $h'(z)$ are required for the risk-averse farmer to use less of the input than the expected profit maximizing farmer? Justify your answer and explain the intuition of your result.
Part III

Answer at most one question from Part III.
Question III.1

Firms 1 and 2 compete as quantity-setting duopolists. Suppose that each firm is restricted to producing only integer values and may choose to set its quantity, $q_i$, equal to 1, 2, 3, or 4 units. Each firm has production costs given by $C(q_i) = 2q_i$. Assume the market inverse demand curve is given by $P(Q) = 8 - Q$, where $P$ is market price and $Q = q_1 + q_2$.

(a) Assume the two firms choose quantity simultaneously. Write the game in normal form (i.e., define a $4 \times 4$ payoff matrix). Find all pure-strategy Nash equilibria for this game.

(b) Does a dominant-strategy equilibrium exist for this game? Explain. If the firms can only choose $q_i$, equal to 1, 2, or 3 units (eliminate the fourth row and fourth column), does a dominant-strategy equilibrium exist? Explain.

(c) Now suppose the firms play the stage game from part (a) in an infinitely repeated game. Assume the discount factor between each period is $\delta$, $0 < \delta < 1$. For what range of discount factors is it possible to support the following strategy as a subgame-perfect equilibrium: in even time periods ($t = 0, 2, 4, 6, \ldots$) $q_1 = 2, q_2 = 1$; in odd time periods ($t = 1, 3, 5, 7, \ldots$) $q_1 = 1, q_2 = 2$; and if some firm deviates from this strategy then both firms revert to producing 2 units every period. Potentially helpful math notes:

$$\sum_{t=0}^{\infty} x\delta^t = \frac{x}{1 - \delta}; \quad \sum_{t=0}^{\infty} x\delta^{2t} = \frac{x}{1 - \delta^2}; \quad \text{and} \quad \sum_{t=1}^{\infty} x\delta^{2t} = \frac{x\delta}{1 - \delta^2}$$

For parts (d) and (e) assume that the firms can choose any positive quantity (they are not restricted to choosing integer values). Assume the market inverse demand curve is given by $P(Q) = A - Q$, where $A$ can take on the values 14 or 26. Assume the probability of each possible value is $1/2$. Suppose firm 1 has done market research and knows the value of $A$ but that firm 2 does not. Each firm still has production costs given by $C(q_i) = 2q_i$.

(d) Solve for Bayesian Nash equilibrium in this game when firms choose strategy simultaneously.

(e) Suppose that after learning whether the value of $A$ is 26 or 14, firm 1 can decide to reveal the value of $A$ to firm 2 or to keep the information private. Solve for the perfect Bayesian equilibrium in the game where in stage 1 firm 1 chooses to reveal information or not when $A = 26$ and when $A = 14$ (note: firm 1 may choose differently depending on the realization of $A$), and in stage 2 the firms choose quantity simultaneously.
Question III.2

Two people jointly produce a product. Let \( E_1 \) be the effort level of person 1 and \( E_2 \) be the effort level of person 2. The amount of production, \( X \), depends on the effort expended by both people: \( X = (E_1 + E_2) + E_1 E_2 \). Assume the price of the product, \( P \), is constant and equal to 1: \( P = 1 \). The two people split the revenue from selling the product, \( PX \), by giving a share \( s_1 \) to person 1 and \( s_2 \) to person 2, with \( s_1 + s_2 = 1 \). Let the cost of effort for person \( i \) be: \( C_i(E_i) = \gamma_i E_i^2 \). Assume the payoff for a person is their share of the revenue minus the cost of their effort.

For parts (a)–(c) assume that \( s_1 = s_2 = 1/2 \), and \( \gamma_1 = \gamma_2 = 2 \).

(a) Solve for a positive (non-zero) Nash equilibrium when the two people choose effort simultaneously.

(b) Solve for the subgame-perfect equilibrium when person 1 chooses effort first and person 2 chooses effort second. How does your answer to this part compare to your answer in part (a)? Explain the intuition for your result.

(c) Solve for the efficient outcome that maximizes the joint outcome. How do the answers in part (a) and (b) compare to the efficient solution? Explain the intuition for your result.

For part (d) continue to assume that \( \gamma_1 = \gamma_2 = 2 \), but not that \( s_1 = s_2 = 1/2 \).

(d) Suppose that the people play a two-stage game. In the first stage they choose effort levels simultaneously. In the second stage person 1 chooses \( s_1 \) and \( s_2 \). Find the subgame-perfect equilibrium for this game.

For part (e) assume that \( s_1 = s_2 = 1/2 \), and \( \gamma_2 = 2 \), but not that \( \gamma_1 = 2 \).

(e) Suppose that person 1 may either be high cost or low cost. Person 1 knows if they are high cost (\( \gamma_1 = 2 \)) or low cost (\( \gamma_1 = 1 \)). Person 2 has beliefs that high cost and low cost are equally likely. Solve for Bayesian Nash equilibrium for the game when the two people choose effort simultaneously.
Part IV

Answer at most one question from Part IV.
Question IV.1

Consider the following $2 \times 2$ competitive exchange economy, with consumers $j = 1, 2$ and goods $x^1$ and $x^2$. The consumers’ preferences are given by

$$U_1(x_1^1, x_1^2) = x_1^1 x_1^2 \quad \text{and} \quad U_2(x_2^1, x_2^2) = -(x_2^1 - 3)^2 - (x_2^2 - 3)^2.$$ 

Endowments are $\omega_1 = (4, 0)$ and $\omega_2 = (0, 4)$.

(a) Find the set of Pareto-optimal allocations for this economy. You do not need to provide a mathematical derivation, though that is a way to get to the solution. Construct a carefully labeled Edgeworth-box diagram depicting the economy, including the endowment, at least one indifference curve for each consumer, and the contract curve.

(b) Derive the offer curves for the two consumers. Now find the competitive-equilibrium price vector and allocation for the economy. Add the offer curves, the equilibrium allocation, and the equilibrium budget line through $\omega$ to your diagram.

(c) Now suppose the consumers’ preferences are as given, but the endowment vector is changed so that $\omega'_1 = (1.5, 0)$ and $\omega'_2 = (2.5, 4)$. Find an equilibrium for this economy, providing both the price vector and the equilibrium allocation. True or false: the equilibrium allocation is Pareto optimal.
Question IV.2

Answer the following two questions regarding exchange economies.

(a) Recall Brouwer’s fixed-point theorem: If $f : S^{n-1} \rightarrow S^{n-1}$ is a continuous function, then there is some $x \in S^{n-1}$ such that $x = f(x)$.

Your task is to provide the proof of the following version of the existence theorem for equilibrium in an exchange economy. Some of the basic assumptions about preferences may be taken for granted, but be sure to specify any crucial particular assumptions that you use. The theorem in question is this:

**Theorem.** (Existence of equilibrium I: strict convexity.) Suppose excess demand $z : S^{n-1} \rightarrow \mathbb{R}^n$ is continuous and that $p \cdot z(p) \equiv 0$ for any $p \in S^{n-1}$. There is $p^* \in S^{n-1}$ such that $z(p^*) \leq 0$.

Provide the proof of this theorem.

(b) In a carefully drawn Edgeworth box, depict an economy that fails to have an equilibrium. Your diagram should include at least one indifference curve for each consumer, an endowment point, and at least one budget line. Be sure to explain clearly how your example relates to the existence theorem.