I. A renewable resource $X$ has a natural growth function $g(X(t))$. There are $N$ resource users. The harvest rate of user $i$ at time $t$ is $h_i \in [0, h]$. The cost per unit of harvest is $c/X_i$. The price of harvest is $P$. Both $c$ and $P$ are constants. The discount rate is $r$.

a. Suppose the resource is an open access resource with free entry ($N \to \infty$). Set up the optimization problem for user $i$. Solve for the value(s) of $X_t$ in equilibrium, $X_t^\infty$.

b. Suppose the resource is a Common Property resource (CPR) with access restricted to a community of $N$ identical users ($N$ is a finite number). Further suppose all $N$ users are cooperative and so the sum of present value of all $N$ users is maximized. Write down the current value Hamiltonian and derive necessary conditions.

c. Denote the steady state value of $X_t$ under CPR regime as $X_t^N$. Assume that the shadow value of $X_t$ is always positive under the CPR regime. Determine the relationship between $X_t^\infty$ and $X_t^N$ ($>, =, <, \geq, \leq$). Show how you come to your conclusion.

d. Suppose the resource is a Common Property resource (CPR) with access restricted to a community of $N$ identical users ($N$ is a finite number). Further suppose all $N$ users are non-cooperative. Write down the current value Hamiltonian for user $i$’s optimization problem and derive necessary conditions. User $i$ is playing a feedback strategy. Denote the steady state value of $X_t$ under CPR regime as $X_t^F$. Is $X_t^F$ larger or smaller than $X_t^N$? Briefly explain your answer.

II. Mike is managing a tree farm. $\theta(t)$ denotes the total tree biomass at time $t$. $\dot{\theta}(t) < 0$.

$\dot{\theta}(t) > 0 \forall \theta(t) \in [0, \bar{\theta})$ and $\dot{\theta}(t) < 0$ if $\theta(t)$ exceeds $\bar{\theta}$. The cost of planting/replanting the trees, which occurs at the beginning of the rotation, is constant at $C$, independent of $\theta(t)$. Trees can be sold at a constant price $P$ per unit of biomass for commercial use. The tree farm is also a hunting farm. There is a continuous flow of hunters to the tree farm that earns Mike an income of $\alpha\theta(t)$ at each instant $t$. $\alpha$ is a constant.

a. Set up Mike’s optimization problem for an infinite horizon.

b. Solve for the equation that characterizes the optimal length of rotation. Interpret this equation in terms of marginal benefit (MB) and marginal cost (MC).

c. Analyze how the following changes affect the optimal length of rotation (that is, lengthen or shorten). You can answer by deriving the comparative statics and looking at the signs. However, simply answering “increase” or “decrease” or “stay the same” and providing intuition will suffice.

i. An increase in $\alpha$ due to the recovery of the economy.

ii. An increase in $r$ due to a change in the Fed’s policy.
III. There are two producers (duopoly) in a resource market, each with reserve $X_i(t)$, $i = 1, 2$. The demand function is $P(t) = P(q(t))$, where $q(t) = q_1(t) + q_2(t)$. $q_1(t)$ and $q_2(t)$ are the extraction rates of producer 1 and 2 at time $t$. The price $P(t)$ decreases in both $q_1(t)$ and $q_2(t)$. The cost per unit of extraction is $c_i[X_i(t)]$, $i = 1, 2$. The discount rate is $r$. Assume that the players continuously observe the (physical) state variables and that, at each instant, they condition their extraction on their own and other players’ reserve levels. The discount rate is $r$.

a. Set up the optimization problem for producer 1.

b. Write down the necessary conditions for the problem you set up in (a) (ignore transversality condition).

c. Now consider the extraction problem from the society’s point of view. The objective is to maximize the social welfare (include both producers and consumers) from consumption, $q(t)$, which is also the extraction rate. Cumulative extraction is $Q(t)$. Stock pollution in the atmosphere is $\alpha Q(t)$. Pollution will NOT cause any damage before the stock pollution, $\alpha Q(t)$, reaches a threshold level, $Z$. $Z$ is a random variable with a probability density function $f(Z)$. Once $Z$ is reached, the society will collapse.

i. Write down the society’s optimization problem. Note welfare is measured using the demand function. At each instant $t$, total welfare of consuming $q(t)$ is:

\[
\int_0^q P(s(t)ds - c_1(X_1)q_1(t) - c_2(X_2)q_2(t), s \text{ is the variable of integration.}
\]

ii. Determine how the uncertainty affects the extraction path. You only need to answer “tilt extraction towards now” or “tilt extraction towards future” and provide some intuition.
IV. A polluting industry consists of two firms that are the only sources of a perfectly mixed pollutant. Before controls are put in place, the firms emit $e_1 = 100$ and $e_2 = 150$. Abatement is costly. The firms’ abatement cost functions are

\[ C_1(q_1) = 3q_1 + \frac{q_1^2}{4} \quad \text{and} \quad C_2(q_2) = 3q_2 + \frac{q_2^2}{8}, \]

where $q_i$ is abatement by firm $i$. An environmental regulator has decided that emissions should be reduced. The regulator’s assessment of abatement benefits is given by

\[ E(B(q)) = 45q - \frac{q^2}{6}, \]

where $q = q_1 + q_2$.

A. Determine the optimal level of the tax that a tax-setting regulator should choose. From this, calculate each firm’s abatement level.

B. Now suppose the regulator wishes to impose a permit-trading scheme to achieve the same level of aggregate abatement. Determine how many permits the regulator will choose to allocate to each of the two firms. If the firms behave competitively in the allowance market, what will be the equilibrium price? How many allowances will be traded?

C. Now suppose firm 1 has market power in the permit market and can select both the permit price and the number of permits that are traded. What will be the outcome in this situation?

V. Two consumers are the only members of an island economy. They have identical preferences over two goods, a private numeraire good $x$ and a pure public good $q$. Preferences are given by

\[ U_i(x_i, q) = \ln x_i + 2 \ln q. \]

Each consumer is endowed with $\omega_i = 10$ of the private good, of which $x_i$ is consumed directly and the remainder $z_i = \omega_i - x_i$ is contributed to the provision of the public good. $q$ is produced according to the simple production function $q = z_1 + z_2$.

A. Determine the outcome $(x_1, x_2, z_1, z_2)$, that a benevolent social planner would choose so as to maximize the unweighted sum of preferences, $W = U_1(x_1, q) + U_2(x_2, q)$.

B. Show that if $z_2$ is held fixed at the solution you found in part a., consumer 1’s best response is to contribute less than $z_1^\ast$.

C. Find the Lindahl price at which the consumers, taking this price as given when selecting their contribution $z_i$, will choose the socially optimal level of contribution to the public good.
A single risk-neutral firm produces output $y$ and emits pollution $x$ into a lake. Let $a$ denote abatement activity and $C(y, a) = 2y^2 + a^2$ the firm’s cost function. Emissions are given by $x = 50 - a + \epsilon$, where $\epsilon$ is a random variable distributed uniformly on the interval $[-10, 10]$. Thus, $E(\epsilon) = 0$. A regulator wishes to achieve the target $\bar{x} = 40$ of pollution. The firm’s output price is $p = 100$. The regulator considers imposing an incentive scheme given by

$$T(x) = \begin{cases} 
    t(x - \bar{x}) + 500 & \text{if } x > \bar{x}; \\
    t(x - \bar{x}) & \text{otherwise},
\end{cases}$$

where $t = 15$ and where $T(x) < 0$ is possible.

A. Determine the firm’s optimal level of output $y$, abatement $a$, and (expected) emissions $x$.

B. What is the probability that $x$ exceeds $\bar{x}$, the regulator’s limit? What is expected profit? What is the firm’s expected level of tax payment?

C. Now suppose that there are two firms, each identical to the firm described above and each emitting pollution into the same lake. The regulator cannot monitor individual emissions but can measure water quality in the lake. The regulator maintains the same target of $\bar{x} = 40$, where $x = x_1 + x_2$, and applies the same incentive scheme to each firm. That is, aggregate emissions determine the value of tax paid by firm $i$:

$$T_i(x_i, x_{-i}) = \begin{cases} 
    t(x - \bar{x}) + 500 & \text{if } x > \bar{x}; \\
    t(x - \bar{x}) & \text{otherwise}.
\end{cases}$$

Assume that “identical” means the firms do not interact strategically. Determine the level of abatement, output, and expected emissions for each firm. Also determine the expected probability that $x$ exceeds $\bar{x}$ and the expected level of aggregate tax payments.