Part I

Answer at most one question from Part I.
Question I.1

A consumer has the following simple indirect utility function for two goods, \( x_1 \) and \( x_2 \):

\[
v(p_1, p_2, w) = \frac{w}{p_1 + p_2}.
\]

(a) Use Roy’s identity to derive the Walrasian demands for \( x_1 \) and \( x_2 \). Are these demands surprising in any way?

(b) Draw a diagram of the consumer’s utility curves in a diagram with \( x_1 \) on the horizontal axis and \( x_2 \) on the vertical axis. You can assume no negative values for either \( x_1 \) or \( x_2 \). Before drawing the diagram, it is useful to compare your answers from part (a) for \( x_1 \) and \( x_2 \) and think about what they imply for the shape of the consumer’s utility curves.

(c) Derive the consumer’s direct utility function. That is, utility as a function of \( x_1 \) and \( x_2 \). It may be helpful to refer to your answer for part (b).

(d) To assure yourself that your answers to (a) and (c) are correct, use those answers to obtain the indirect utility function, and check whether it is the same indirect utility function that is given at the beginning of this problem.

(e) Finally, derive the Hicksian demands for \( x_1 \) and \( x_2 \).
Question I.2

Consider the following direct utility function in a situation with two goods, $x_1$ and $x_2$:

$$u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$$

(a) Assuming that $w$ equals wealth, and that the consumer spends his or her entire wealth on goods $x_1$ and $x_2$, use constrained optimization to solve for the Walrasian demands for $x_1$ and $x_2$.

(b) Use your answers to (a) to obtain the indirect utility function and the expenditure function that correspond to the original utility function.

(c) Consider an initial situation where $p_1 = 1$, $p_2 = 1$ and $w = 100$. Suddenly $p_1$ increases to 2 ($p_1 = 2$). How much additional money does the consumer need to be given so that he or she has the same utility that he or she had before the $p_1$ increased? The answer should be a function of $\alpha$ and some numbers.

(d) Is your answer to part (c) the equivalent variation (EV), the compensating variation (CV), or something else? Explain your answer.
Part II

Answer at most one question from Part II.
Question II.1

Consider the production possibility set (PPS):

$$\text{PPS} = \left\{ (q, -z) \in \mathbb{R}_+^2 \times \mathbb{R}_-^2 : z_1^{\alpha_1} z_2^{\alpha_2} \geq q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}} \right\},$$

where $q_1$ and $q_2$ are outputs; $z_1$ and $z_2$ are inputs; and $\alpha_1 > 0$ and $\alpha_2 > 0$ are constant parameters. This PPS is nonempty, strictly convex, closed, and satisfies weak free disposal in inputs and outputs.

(a) Derive the input distance function assuming $q_1 > 0$ or $q_2 > 0$. What homogeneity property will this input distance function satisfy? Show that this property is in fact satisfied.

(b) Assuming $\alpha_1 = \alpha_2$, the input distance function for this PPS will yield the cost function

$$C(r, q) = 2(r_1 r_2)^{\frac{1}{2}} \left( q_1^{\frac{3}{4}} + q_2^{\frac{1}{4}} \right),$$

where $r_1 > 0$ and $r_2 > 0$ are the input prices. Use this cost function to derive the conditional input demands.

(c) Let $p_1 > 0$ and $p_2 > 0$ be the competitive price of $q_1$ and $q_2$. Using the cost function in (b), find the profit-maximizing unconditional supplies. Also, derive the unconditional input demands. (Hint: This can be accomplished most rapidly using duality results.)

(d) It is easy to verify that the unconditional supplies in (c) are non-decreasing in their own price. Show that this result holds in general for a profit function derived from a production possibility set with $N$ inputs and $M$ outputs.
Question II.2

Consider the uncertain revenue function

\[ R(z_1, z_2) = \mu(z_1) + \sigma(z_2)\varepsilon, \]

where \( z_1 \geq 0 \) and \( z_2 \geq 0 \) are inputs, \( \mu(z_1) > 0 \) is expected revenue given \( z_1 \), \( \sigma(z_2) > 0 \) is the standard deviation of revenue given \( z_2 \), and \( \varepsilon \) is a random shock to revenue. Assume \( \mu'(z_1) > 0, \mu''(z_1) < 0, \sigma'(z_2) < 0, \sigma''(z_2) > 0, E(\varepsilon) = 0 \), and \( E(\varepsilon^2) = 1 \).

There are two types of firms. One maximizes its expected profit. The other is risk averse and maximizes the expected utility of profit where the strictly increasing and strictly concave function \( u(\cdot) \) characterizes risk preferences. Let \( r_1 > 0 \) and \( r_2 > 0 \) be the competitive prices of the inputs.

(a) Set up the profit-maximizing firm’s optimization problem and derive its first-order conditions. (Remember your non-negativity constraints.) What condition (if any) will insure that \( z_1 > 0 \) at the optimum? What condition (if any) will insure that \( z_2 > 0 \) at the optimum? Explain the economic intuition of your results.

(b) Set up the expected utility-maximizing firm’s optimization problem and derive its first-order conditions. (Remember your non-negativity constraints.) What condition (if any) will insure that \( z_1 > 0 \) at the optimum? What condition (if any) will insure that \( z_2 > 0 \) at the optimum? Explain the economic intuition of your results.

(c) Compared to the profit-maximizing firm, will the expected utility-maximizing firm use more, less, or the same amount of \( z_1 \)? Will the expected utility-maximizing firm use more, less, or the same amount of \( z_2 \)? Justify your answers.
Part III

Answer at most one question from Part III.
Question III.1

Gigantron Stores and MegaMart compete as duopolists in selling consumer products at the retail level. Each firm buys consumer products at wholesale for a cost of $2c$ per unit. Consumers have some brand loyalty to the firms so that products from each firm are viewed as imperfect substitutes. Assume that Gigantron has a demand curve:

\[ X(P_x, P_y) = a - (1/2)P_x + bP_y, \]

where \( X \) is the quantity demanded from Gigantron, \( P_x \) is the price of a unit of Gigantron consumer products and \( P_y \) is the price of a unit of Megamart consumer products, and \( a > 0, b > 0 \). Assume that Megamart has a demand curve given by

\[ Y(P_x, P_y) = \alpha - (1/2)P_y + \beta P_x, \]

with \( \alpha > 0 \) and \( \beta > 0 \). Assume that \( \beta b < (1/2) \).

(a) Solve for the Bertrand equilibrium in which both firms simultaneously choose price.

(b) Now suppose that Gigantron gets to move first and set price prior to Megamart. Solve for subgame-perfect equilibrium prices.

(c) Are equilibrium prices higher, lower, or the same in part (b) as compared to part (a)? Explain the economic intuition for this result.
Question III.2

Firm X hires employees to work at its facility. Potential employees are of two types: high-productivity types \((t = H)\) and low-productivity types \((t = L)\). Assume that \(p(t = H) = 0.25\) and \(p(t = L) = 0.75\). Each potential employee chooses an education level \(e \geq 0\). The cost of education is \(c(e, H) = 2e^2\) for high types and \(c(e, L) = 3e^2\) for low types. If employed, high productive types produce output of \(y(e, H) = 2e\) while low productive types produce output of \(y(e, L) = e\). Each unit of output is worth $6 to Firm X. Firm X observes the education level for a potential employee but does not observe the type. Firm X is one of many identical employers and must pay competitive wages—workers are paid wages equal to the expected value of output they create and Firm X earns expected profit of zero. The payoff for an employee hired by Firm X is the wage minus the cost of education: 
\[ U_E = w(e) - c(e, t), \text{ for } t = H \text{ or } L. \]

(a) What is the Pareto-optimal education level for each type?

(b) Is it possible for Firm X to offer competitive wages and obtain the optimal education level in a perfect Bayesian equilibrium? Explain.

(c) Solve for at least one separating perfect Bayesian equilibrium. Show that your solution satisfies the conditions for perfect Bayesian equilibrium.

(d) Solve for at least one pooling perfect Bayesian equilibrium. Show that your solution satisfies the conditions for perfect Bayesian equilibrium.
Part IV

Answer at most one question from Part IV.
Question IV.1

Consider an exchange economy consisting of two consumers and two goods, $x^1$ and $x^2$. The aggregate endowment is $\omega = (12, 6)$ and the consumers’ utility functions are $U_1(x^1_1, x^2_1) = 2x^1_1 + x^2_1$ and $U_2(x^1_2, x^2_2) = x^1_2 x^2_2$ respectively.

(a) Derive the contract curve for this economy. That is, find all Pareto-optimal allocations. You may choose whether to perform the required mathematical derivation or to construct a carefully labeled Edgeworth-box diagram to explain your answer. Either can be a suitable response.

(b) Now suppose the individual endowments are $\omega_1 = (6, 6)$ and $\omega_2 = (6, 0)$. Derive the offer curves for the two consumers. Find the Walrasian equilibrium allocation and prices $(x^*, p^*)$.

(c) Let $z(p)$ denote aggregate excess demand. Show that Walras’s law holds for this economy: for any $p \in \mathbb{R}_{++}^n$, $p \cdot z(p) = 0$.

(d) State the first welfare theorem, being careful to say clearly which assumptions are needed in order for the result to hold in general. Show that the first welfare theorem holds for this economy.
Question IV.2

Consider Arrow’s social-choice problem, with individuals $J = \{1, \ldots, n\}$, alternatives $X$, individual orderings $P_j$ over $X$, and the SWF $f(\{P_j\}_{j=1}^n)$. Recall that the set of individuals $D \subset I$ is semi-decisive for alternatives $x$ and $y$ if $[\forall j \in D, xP_j y]$ and $[\forall j \notin D, yP_j x]$ together imply $[xP y]$. Recall also that the the set of individuals $D \subset I$ is decisive for alternatives $x$ and $y$ if $[\forall j \in D, xP_j y]$ implies $[xP y]$.

Provide a proof of the second step of Arrow’s theorem. Start with the result of the first step, that there is some pair $a, b \in X$ for which some person $j$ is semi-decisive. Show that this person $j$ is decisive over all $x, y \in X$ and so $j$ is a dictator.