Ph.D. Preliminary Examination
MICROECONOMIC THEORY
Applied Economics Graduate Program
May 2012

The time limit for this exam is 4 hours. It has four sections. Each section includes two questions. You are to answer one question from each section. Before turning in your exam, please number each page of your answers with the section and question number (for example, III.2 for section III and question 2). Indicate, by circling the appropriate selections below, which questions you have completed. If you answer more than one question in a section, and do not circle a question corresponding to that section, the first of the two that appears in your solution paper will be graded.

Section I: Question 1 Question 2
Section II: Question 1 Question 2
Section III: Question 1 Question 2
Section IV: Question 1 Question 2

NOTE: The exam should have 13 pages including this cover page.
Part I

Answer at most one question from Part I.
Question I.1

Consider a simple utility function for two goods, \(x_1\) and \(x_2\):

\[
u = (x_1 - \gamma_1)^{\beta_1} (x_2 - \gamma_2)^{\beta_2}, \quad \text{with } \beta_1 > 0, \beta_2 > 0.
\]

The parameters \(\gamma_1\) and \(\gamma_2\) are constants, but they could be positive or negative.

(a) Someone suggests that “it would be okay” to assume that \(\beta_1 + \beta_2 = 1\). Yet someone else claims that this restriction may be unrealistic. Which person is correct? Briefly explain your answer.

(b) Suppose that the person with the above utility function is struck by lightning. He survives, except now his utility function is \(u = \beta_1 \log(x_1 - \gamma_1) + \beta_2 \log(x_2 - \gamma_2)\). Will his consumption decisions change as a result of being struck by lightning? Briefly explain your answer.

(c) Returning to the original utility function, assume that the consumer faces the budget constraint \(p_1x_1 + p_2x_2 = w\), where \(p_1 > 0, p_2 > 0\) are prices and \(w > 0\) is wealth. Set up the Lagrangean and solve for the two first-order conditions. Then use the budget constraint to solve for the Walrasian demands of both goods. These should be functions of \(p_1, p_2, w, \beta_1, \beta_2, \gamma_1,\) and \(\gamma_2\). Finally, use your answer to part (a) to simplify your answer.

(d) Finally, turn to a more general utility function with \(L\) goods:

\[
u = \prod_{k=1}^{L} (x_k - \gamma_k)^{\beta_k}, \quad \text{where } \beta_k > 0 \text{ for all } k,
\]

which is maximized subject to the budget constraint \(\sum_{k=1}^{L} p_k x_k = w\). Using the same approach as in part (c), derive the Walrasian demands for the \(L\) goods. (Hint: first substitute out the Lagrangean multiplier using the first-order conditions for two goods, \(i\) and \(j\), then find an expression for \(p_i x_i\) and sum that expression over all \(i\), using a normalization similar to the one used in part (c)).
Consider an expenditure function that has the form
\[
\log[e(p, u)] = \sum_{k=1}^{L} \alpha_k \log(p_k) + u \prod_{k=1}^{L} p_k^{\beta_k} .
\]
Don’t worry that this is expressed as \(\log[e(p, u)]\) instead of as \(e(p, u)\). After all, if we apply the \(\exp(\cdot)\) function to both sides, the left-hand side will become \(e(p, u)\).

(a) Apply Shephard’s lemma to this expenditure function to obtain the Hicksian demand for good \(i\). (Hint: Differentiate with respect to \(\log(p_i)\) to obtain an elasticity that includes the Hicksian demand. Note also that \(p_k^{\beta_k} = (e^{\log(p_k)})^{\beta_k}\). The Hicksian demand should be a function of the \(\alpha\)'s, the \(\beta\)'s, prices, and \(u\).)

(b) Derive the indirect utility function that is associated with this cost function. This is easier than part (a).

(c) Finally, use your answer to (b) to obtain the Walrasian demands. What is the name of the derivation that is used to obtain the Walrasian demands?
Part II

Answer at most one question from Part II.
Question II.1

Consider a firm that produces a single output \( q \geq 0 \) using inputs \( z_1 \geq 0 \) and \( z_2 \geq 0 \), where the input-requirement set is nonempty, strictly convex, closed, and satisfies weak free disposal. Assume the firm operates in competitive markets. The firm’s profit function is

\[
\pi(r_1, r_2, p) = \frac{p^\alpha}{4(r_1 + \sqrt{r_1r_2 + r_2})},
\]

where \( p > 0 \) is the price of output, \( r_1 > 0 \) and \( r_2 > 0 \) are the input prices, and \( \alpha \) is a constant parameter.

(a) What condition on \( \alpha \) (if any) is required for \( \pi(r_1, r_2, p) \) to satisfy the price homogeneity property of a valid profit function? Justify your answer.

(b) Use \( \pi(r_1, r_2, p) \) to derive the firm’s unconditional supply and factor demands.

(c) Derive the firm’s conditional factor demands and cost function.

(d) It is easy to verify that the conditional input demands in (c) are non-increasing in their own price. Show that this result holds in general for a cost function derived from a production possibility set with \( N \) inputs and \( M \) outputs.
Question II.2

Consider a firm that produces output $q \geq 0$ at a cost of $c(q)$, where $c'(q) > 0$ and $c''(q) > 0$. Also assume that there is a probability $1 > \alpha > 0$ that the firm experiences an equipment failure and incurs additional repair costs equal to $c_R > 0$ per unit of output or $c_R q$ in total. The competitive price of output is $p > 0$.

(a) Derive the firm’s first-order condition for an interior solution assuming its objective is to maximize expected profit. What is the economic intuition of this condition?

(b) Derive the firm’s first-order condition for an interior solution assuming its objective is to maximize its expected utility of profit, where the strictly increasing and strictly concave function $u(\cdot)$ characterizes its risk preferences.

(c) Will the firm produce more if its objective is to maximize expected profit or if its objective is to maximize the expected utility of profit? Justify your answer and provide the economic intuition for your result.
Part III

Answer at most one question from Part III.
Question III.1

There are \( N > 2 \) profit-maximizing firms that compete as Cournot oligopolists in a market with inverse demand given by \( P(Q) = a - Q \), where \( P \) is price, \( Q = \sum_{i=1}^{N} q_i \) and \( q_i \) is the output of firm \( i \) for \( i = 1, \ldots N \). Assume that all firms have the same cost function: \( C(q_i) = c q_i \).

(a) Solve for Cournot equilibrium. Show that the profit for each firm in Cournot equilibrium, \( \pi^C_i \), is equal to the square of output: \( \pi^C_i = q_i^2 \).

(b) Suppose that prior to choosing quantities, two firms have the option to merge into a single firm. Compare the profit of the single merged firm in this new Cournot equilibrium, with a total of \( N - 1 \) firms, to the sum of the two firms’ individual profits in the original Cournot equilibrium, where there were \( N \) firms. Show whether the firms should merge or not.

(c) Now suppose \( N \) firms compete in an infinitely repeated game in which in each period they simultaneously choose \( q_i \). Let \( \delta \) be the discount factor between periods \((0 < \delta < 1)\). Let \( t = 0, 1, 2, \ldots \) represent the time period and let \( Q^M \) denote the monopoly level of output. Suppose that each firm plays the following simple trigger strategy:

- Set \( q_i = Q^M / N \) in period \( t = 0 \) and in all \( t > 0 \) for which aggregate output was \( Q^M \) in all preceding periods, and
- Set \( q_i \) equal to the Cournot equilibrium output for all \( t > 0 \) for which aggregate output was not \( Q^M \) in all preceding periods.

For what range of the discount factor \( \delta \) can the trigger strategy support collusion at the monopoly output level as a subgame-perfect equilibrium outcome?
Question III.2

A painting is to be auctioned in a sealed-bid auction in which all bidders simultaneously submit a bid and the highest bid wins the painting. Suppose there are only two bidders, \( i = 1, 2 \). Let \( b_i \) be the bid of player \( i \). The value of owning the painting for bidder \( i \) is \( v_i \). Assume that each \( v_i \) is drawn from a uniform distribution on \([0, 1]\). Each bidder knows her own value but not the other bidder’s value.

(a) In a first-price sealed-bid auction the bidder with the highest bid wins the painting and pays her bid to the auctioneer. Show that bidding \( b_i = v_i / 2 \) is a Bayesian Nash equilibrium.

(b) In a second-price sealed-bid auction the bidder with the highest bid wins the painting and pays the bid of the second-highest bid to the auctioneer. Show that it is a dominant strategy for each bidder to bid \( b_i = v_i \).

(c) The auctioneer’s goal is to maximize the expected revenue from the auction. Show whether the auctioneer should choose the first-price sealed-bid auction or the second-price sealed-bid auction, or whether the auctions generate the same expected revenue. (Hint: A potentially useful mathematical fact is that the expected value of the maximum of two independent random variables uniformly distributed on \([0, x]\) is \(2x/3\).)
Part IV

Answer at most one question from Part IV.
Question IV.1

Consider an exchange economy with two consumers, \( j = 1, 2 \), and two goods \( x^1 \) and \( x^2 \). Consumer 1 has an endowment of \( \omega_1 = (6, 0) \) and consumer 2 has an endowment of \( \omega_2 = (0, 6) \). Subscripts denote consumers and superscripts denote goods. Utility functions are given by

\[
U_1(x_1^1, x_1^2) = \begin{cases} 
  x_1^1 x_1^2 & \text{if } x_1^2 \geq 2x_1^1 \\
  x_1^1 + x_1^2 & \text{if } x_1^1/2 < x_1^2 < 2x_1^1 \\
  x_1^1 x_1^2 & \text{if } x_1^2 \leq x_1^1/2
\end{cases}
\]

and

\[
U_2(x_2^1, x_2^2) = x_2^1 x_2^2.
\]

(Consumer 1’s indifference curves have a flat portion between the rays along which \( x_1^2 = 2x_1^1 \) and \( x_1^2 = x_1^1/2 \).)

(a) Derive the offer curves for the two consumers. Solve for a Walrasian equilibrium allocation and prices \((x^*, p^*)\). You may include a carefully labeled diagram as part of your answer if you wish, but this is not required.

(b) Derive the contract curve for this economy. (This is the set that Mas-Colell calls the Pareto set.) Prove that the Walrasian equilibrium allocation you found in part (a) is Pareto optimal.

(c) Consider an endowment vector \( \tilde{\omega} \in \mathbb{R}^4_+ \) with the property that the aggregate endowment is the same as for \( \omega \): for each good \( i \), \( \tilde{\omega}_i^1 + \tilde{\omega}_i^2 = 6 \). Derive expressions for (i) the associated equilibrium price vector \( \tilde{p}(\tilde{\omega}) \) and (ii) the associated equilibrium allocation \( \tilde{x}(\tilde{\omega}) \).
Question IV.1

Consider a 2-person economy in the form of Bergstrom, Blume, and Varian with one private good $x$ and one public good $G$. Each consumer is endowed with a quantity $w_i$ of the private good, of which $x_i$ is consumed and $g_i$ is contributed to the linear provision of the public good. Thus, $w_i = x_i + g_i$ and $G = g_1 + g_2$. Utility functions are Cobb-Douglas, given by

$$U_1(x_1, G) = \ln x_1 + \ln G$$ and $$U_2(x_2, G) = \ln x_2 + 2 \ln G.$$ 

(a) Suppose the initial endowments are $\bar{w}_1 = 10$ and $\bar{w}_2 = 5$. Solve for the optimum given by Samuelson’s condition.

(b) Solve each consumer’s problem to obtain the best-response functions $g_i(g_{-i})$ that lead to the voluntary-contribution equilibrium, where $g_{-1} = g_2$ and $g_{-2} = g_1$. In BBV’s notation the best-response functions are of the form

$$g_i = \max\{f_i(w_i + g_{-i}) - g_{-i}, 0\},$$

where $f_i(w_i, g_{-i})$ is the demand for $G$ that arises from the solution to $\max_{x_i, G} U_i(x_i, G)$ subject to $x_i + G = w_i + g_{-i}$.

(c) Suppose that initially $\bar{w} = (10, 5)$ as in part (a). Solve for the $g_i^*, G^*$, and the $x_i^*$. Find the range of $w_2$ such that $w_1 + w_2 = 15$ and $G^*$ remains at the level you found for $\bar{w}$.

(d) Now suppose that both endowments increase proportionately, so that $w_i' = t\bar{w}_i$ for $t > 1$. Show that the new equilibrium level of the public good, $G'$, is such that $G' > G^*$. 