# PRODUCTION

**ECON 8001-2** **Instructor: Terry Hurley**

## INTRODUCTION (MWG Chapter 5 & Varian Chapter 1-6)

Having completed an exhausting exploration of consumer behavior and demand, it is time to consider producer behavior and supply. To start, we will discuss the basic idea of combining commodities to produce new commodities, which is basically the notion of production functions. We will then consider how producers determine how to produce and how much to produce. We will look at these questions from two perspectives: cost minimization and profit maximization. With these two perspectives, we will develop the concepts of cost functions, conditional factor demand, profit functions, supply, and unconditional factor demand. We will discuss the properties of these objects and how they are related. Our ultimate goal is the careful derivation of the supply side of our market economy.

## NOTATION

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| $L$: | Number of commodities including those that must be produced from other commodities (e.g. cloth and shirts) as well as primary factors (e.g. land and labor). |
| $y \in \Re^L$: | Production vector. |
| $y_l$: | Particular element of a production vector ($y_l > 0$ for a net output, $y_l < 0$ for a net input, and $y_l = 0$ if not used in production). |
| $q \in \Re^M_+$: | Vector of outputs. |
| $q_m \geq 0$: | Specific output. |
| $z \in \Re^N_+$: | Vector of factors/inputs. |
| $z_n \geq 0$: | Specific factor/input. |
| $p \in \Re^L_{++}$: | Vector of prices. |
| $p \in \Re^M_{++}$: | Vector of output prices. |
| $r \in \Re^N_{++}$: | Vector of factor/input prices. |

## PRODUCTION

Most of you are likely familiar with production in the context of a single output and many factors where there is a clear delineation between what is an output and what is a factor or input. We will start here and then work our way to more general characterizations. While working through the classical single output and many factors case, it is helpful to pay close attention to the similarities and differences between this theory and the theory of consumer behavior.

We will begin with the notion of the *Production Possibilities Set*. This set represents all feasible combinations of outputs and inputs. We will label this Production Possibilities Set $Y$. In general, $Y \subset \Re^L$. For our single output and many input case, $Y \subset \Re^1_+ \times \Re^N_-$. To be clear on how we are doing things here, $y = (q, -z)$ where output is measured as a positive number and inputs are measured as negative numbers in this set. If $y$ is to be an option for the firm, $y \in Y$.

The most talked about factor determining $Y$ in general is technology, but it is also reasonable to think about production possibilities being constrained by institutional factors. Another common distinction to talk about in terms production possibilities is time. Inputs like labor can often be immediately reallocated, while capital inputs like buildings and machinery can take much longer to reallocate. This leads to the common distinction between ***Long-Run*** and ***Short-Run Production***. In the Short-Run, some inputs cannot be reallocated, which further restricts a producer's production possibilities. In the Long-Run, all inputs can be reallocated, which gives a producer as much flexibility as possible. Of course the black and white distinction between Long-Run and Short-Run production is really grey and typically determined by the application. For now, we do not need to be too careful with this distinction.

With the Production Possibilities Set defined, we can define a couple of other concepts, some of which should be familiar.

DEFINITIONS

*Input Requirements Set* ($Z(q)$): All combinations of inputs capable of producing a particular level of output: $Z(q) = \{ z \in \Re^N_+ : (q, -z) \in Y\}$.

*Isoquant* ($Q(q)$): All combinations of inputs capable of producing a particular level of output, but not more: $Q(q) = \{z \in \Re^N_+ : z \in Z(q) \text{ and } z \notin Z(q') \text{ for all } q' \geq q \text{ and } q' \neq q\}$.

*Technologically Efficient*: A production vector is technologically efficient if there is no way to produce more output with the same amount of inputs: $(q', -z') \in Y$ such that $q' \geq q$ for all $q$ where $(q, -z') \in Y$.

*Transformation Function* ($T(y)$): A real valued function $T$: $\hat{A}^L \rightarrow \hat{A}$ where $T(y) = 0$ if and only if $y$ is technologically efficient.

These definitions are written in their general form. That is, we can think of $q$ as being a vector of outputs. With a single output and many factors, the Transformation Function is usually written in the form of a Production Function: $q = f(z)$, which would correspond to $T(y) = T(q, -z) = q - f(z)$.

The Input Requirement Set is directly analogous to the Upper Contour Set we defined for the consumer's problem, while an Isoquant is directly analogous to the Indifference Contour Set. The Transformation or Production Function is directly analogous to the utility function. There is however an important difference between the Production Function and Utility Function. In both cases, the inputs into each are directly observable. The difference is that the Production Function is cardinal, while the utility function is only ordinal. Why is this the case? It is the case because output is typically directly observable, while an individual's level of satisfaction is not.

With the consumer problem, we introduced a variety of assumptions regarding individual preferences, which allowed us to say quite a bit in terms how people might behave in a market environment. We can do a similar exercise with production.

DEFINITIONS:

*Monotone Production Possibilities Set*: A Production Possibilities Set is monotone if for all $(q, -z), (q, -z') \in Y$, if $z \in Z(q)$ and $z' \geq z$, then $z' \in Z(q)$, or if for all $(q, -z), (q', -z) \in Y$, if $z \in Z(q')$ and $q' \geq q$, $z \in Z(q)$.

*Free Disposal*: If $y \in Y$ and $y' \leq y$, then $y' \in Y$ or $Y - \Re_+^L \subset Y$.

This monotonicity definition implies that adding more factors cannot decrease output, which seems to make sense assuming the costless disposal of the inputs we do not want to use. Free disposal is just the generalization of this idea to production possibility sets where commodities can be inputs or outputs. The definitions are analogous to the definition of monotonicity for preference relations.

*Convex Factor Requirement Set*: A Input Requirement Set is convex if for all $z, z' \in Z(q)$ and all $a \in [0, 1]$, $az + (1 - a)z' \in Z(q)$.

*Strictly Convex Factor Requirement Set*: An Input Requirement Set is strictly convex if for all $z$, $z' \in Z(q)$, $z \neq z'$ and all $a \in (0, 1)$, $az + (1 - a)z' \in Z(q)$ and $az + (1 - a)z' \notin Q(q)$.

*Convex Production Possibilities Set*: A Production Possibilities Set is convex if for all $y, y' \in Y$ and all $a \in [0, 1]$, $ay + (1 - a)y' \in Y$.

The first of these convexity definitions says that if we can produce $q$ with two different vectors of factors, then we can also produce $q$ with any linear combination of the two vectors. The second says that if we can produce $q$ with two different vectors of factors, then we can produce more than $q$ with any linear combination of the two vectors that is not equal to either. The third is just a generalization to the case of many outputs and inputs. Intuitively, convexity implies that inputs will exhibit diminishing marginal returns or that balanced input combinations are more productive than unbalanced ones. Note that if the Production Possibility Set is convex, then the Input Requirement Set will also be convex. The definitions are analogous to the convexity definitions for preference relations.

There are some additional technical assumptions we will make use of on occasion to make our life easier (some with greater intuitive appeal than others):

*Regular Factor Requirement Set*: An Input Requirement Set is regular if it is closed and nonempty.

*Nonempty Production Possibilities Set*: $Y \neq \emptyset$.

*Closed Production Possibilities Set*: A Production Possibilities Set is closed if $y^n \to y$ and $y^n \in Y$ for all $n$, then $y \in Y$.

These three definitions provide technical restrictions that ensure there is something for the producer to do and something for the producer to do optimally.

*No Free Lunch*: If $y \in Y$ and $y \geq 0$, then $y = 0$ or if $z = 0$, then $q = 0$.

No free lunch says we cannot produce outputs if we do not use any factors.

*Additivity* (*Free Entry*): If $y, y' \in Y$, then $y + y' \in Y$ or if $z \in Z(q)$ and $z' \in Z(q')$, then $z + z' \in Z(q + q')$.

Additivity says that if there are two different ways to produce, then there is a third way to produce that combines the two.

There are a variety of terms and quantities that have been developed to help describe production relationships. Many of these should be familiar.

DEFINITIONS

*Technical Rate of Substitution/Marginal Rate of Technical Substitution* (*MRTS*): The amount by which one factor of production must increase in order to maintain the same level of production when another factor decreases assuming production is efficient.

The MRTS is analogous to the Marginal Rate of Substitution in the world of consumers. Graphically, it represents the slope of an Isoquant. Mathematically, if we have the production function $q = f(z)$ and we want to find the MRTS between factor $l$ and $k$, we can differentiate with respect to $l$ and $k$ to get 0 = $\frac{\partial f(z)}{\partial z_l}dz_l + \frac{\partial f(z)}{\partial z_k}dz_k$, which after rearranging yields

**P1**
$$\frac{dz_l}{dz_k} = -\frac{\frac{\partial f(z)}{\partial z_k}}{\frac{\partial f(z)}{\partial z_l}}.$$

Often, the minus sign is dropped to yield the MRTS of $\frac{\frac{\partial f(z)}{\partial z_k}}{\frac{\partial f(z)}{\partial z_l}}$. This notion can be generalized with respect to the transformation function by differentiating $T(y) = 0$ with respect $y_k = -z_k < 0$ and $y_l = -z_l < 0$: $\frac{\partial T(y)}{\partial y_k}dy_k + \frac{\partial T(y)}{\partial y_l}dy_l$ = 0 or $-\frac{\partial T(y)}{\partial y_k}dz_k - \frac{\partial T(y)}{\partial y_l}dz_l$ = 0. Rearranging then yields

**P1'**
$$\frac{dz_l}{dz_k} = -\frac{\frac{\partial T(y)}{\partial y_k}}{\frac{\partial T(y)}{\partial y_l}},$$

which is analogous to equation P1 except we are holding all outputs fixed.

When there are multiple outputs, there is another quantity of interest:

*Marginal Rate of Transformation* (*MRT*): The amount by which one output must decrease, while another output increases in order to maintain efficient production with the same level of factors.

For the transformation function $T(y) = 0$, the MRT for $y_k > 0$ and $y_l > 0$ can again be found by differentiating: $\frac{\partial T(y)}{\partial y_k}dy_k + \frac{\partial T(y)}{\partial y_l}dy_l = 0$ or

**P2** $$\frac{dy_l}{dy_k} = -\frac{\frac{\partial T(y)}{\partial y_k}}{\frac{\partial T(y)}{\partial y_l}}.$$

Another common way to characterize production functions is in terms of what is referred to as returns to scale:

*Nonincreasing Returns to Scale*: A Production Possibilities Set exhibits Nonincreasing Returns to Scale if any feasible production vector can be scaled down: if $y \in Y$, then $ay \in Y$ for all $a \in [0, 1]$ or $tf(z) \geq f(tz)$ for all $t > 1$.

*Nondecreasing Returns to Scale*: A Production Possibilities Set exhibits Nondecreasing Returns to Scale if any feasible production vector can be scaled up: if $y \in Y$, then $ey \in Y$ for any $e \geq 1$ or $f(tz) \geq tf(z)$ for all $t > 1$.

*Constant Returns to Scale*: A Production Possibilities Set exhibits Constant Returns to Scale if any feasible production vector cannot be scaled up or down: if $y \in Y$, then $ty \in Y$ for all $t \geq 0$ or $f(tz) = tf(z)$ for all $t > 0$.

The concept of returns to scale has important implications in terms of industry structure. We will not go into these implications in detail here, but will note that Nondecreasing Returns to Scale is often used as an explanation for industries with a few very large firms, while Nonincreasing Returns to Scale is often used as an explanation for industries with lots of very small firms. Constant Returns to Scale is often associated with industries where firm size varies substantially.

The definitions of returns to scale above are global in nature. It is often reasonable to believe that a production process exhibits Nondecreasing Returns to Scale for some levels of production and Nonincreasing Returns to Scale for other levels of production. A more local characterization of returns to scale can be obtained using the *Elasticity of Scale*:

**P3** $$e(z) = \frac{df(ez)}{de}\frac{e}{f(ez)} \text{ where } e = 1.$$

There is an important relationship between this Elasticity of Scale and the *Output Elasticities* of our factors of production, $e_l(z) = \frac{\partial f(z)}{\partial z_l}\frac{z_l}{f(z)}$. That relationship is $e(z) = \sum_{l=1}^{L} e_l(z)$. Intuitively, the Elasticity of Scale tells us the percentage change in output for a one percent increase in all factors, while the Output Elasticity of a factor tells us the percentage change in output for a one percent increase in the factor of interest. Again, all this assume production is technically efficient.

Before moving on to how producers might choose how to produce and how much to produce, it is worth talking about a special class of production technologies that have been widely studied.

DEFINITION:

*Homothetic Function*: A function that is a monotonic transformation of a function that is homogeneous of degree 1: $f(z) = g(h(z))$ where $g(\cdot)$ is an increasing monotonic function and $h(z)$ is homogenous of degree 1 in $z$.

Homothetic Production Functions satisfy a several useful properties that make them easy to work with:

(i) If $f(z^0) = f(z^1)$ for $z^0, z^1 \in \Re^L_+$, then $f(az^0) = f(az^1)$ for all $a > 0$ and

(ii) $\dfrac{\frac{\partial f(z)}{\partial z_l}}{\frac{\partial f(z)}{\partial z_k}} = \dfrac{\frac{\partial f(az)}{\partial z_l}}{\frac{\partial f(az)}{\partial z_k}}$ for all $l$ and $k$ and all $a > 0$.