

EXPENDITURE MINIMIZATION

ECON 8001-2

Instructor: Terry Hurley

INTRODUCTION

The utility maximization problem frames individual decision making in a natural way: the individual has a budget and can use it to buy things. People commonly face these types of decisions in day to day life. However, there is another way to frame the problem that is also quite natural if one thinks about planning for the future: if I want maintain my current standard of living in retirement, what is the least amount of wealth I need? This second way of framing the problem is referred to as the expenditure minimization problem and as we will see, its solution will provide new insights on its own and when combined with the insights of the utility maximization problem.

THE SETUP

Everything in the expenditure minimization problem (*EMP* for short) is the same as with the utility maximization problem except we need to introduce u , which represents some minimal level of utility (or standard of living) an individual would like to achieve. I will also replace the x 's used in the *UMP* problem with h 's (i.e. $h \in X = \mathfrak{R}^L_+$) to emphasize the distinction between the two problems. We can now define the expenditure minimization problem as

$$\mathbf{EM1} \quad h(p, u) = \{h \in X: u(h) \geq u \text{ and } p \cdot h' \geq p \cdot h \text{ for all } h' \text{ where } u(h') \geq u \text{ and } h' \in X\}.$$

or

$$\mathbf{EM1}' \quad \min_{h \geq 0} p \cdot h \text{ such that } u(h) \geq u.$$

The Lagrangian for *EMP* can be written as

$$\mathbf{EM2} \quad L = p \cdot h + g(u - u(h)).$$

The first-order necessary conditions for this problem are

$$\mathbf{EM3} \quad \frac{\partial L}{\partial h_l} = p_l - g^* \frac{\partial u(h^*)}{\partial h_l} \geq 0, \quad \frac{\partial L}{\partial h_l} h_l^* = 0, \text{ and} \quad h_l^* \geq 0 \text{ for } l = 1, \dots, L, \text{ and}$$

$$\mathbf{EM4} \quad \frac{\partial L}{\partial g} = u - u(h^*) \leq 0, \quad \frac{\partial L}{\partial g} g^* = 0, \text{ and} \quad g^* \geq 0.$$

Using matrix notation, **EM3** can also be written as

$$\mathbf{EM3}' \quad p - g^* \nabla u(h^*) \geq 0, \quad h^* \cdot [p - g^* \nabla u(h^*)] = 0, \text{ and} \quad h^* \geq 0.$$

To be sure we actually find a global minimum, we should check the second-order conditions. For now however, we will just note that these conditions will be satisfied if we maintain our assumptions of rationality, local nonsatiation, strict convexity, and continuity. The solution to this problem is a demand function: $h^* = h(p, u)$. If $h_l^* > 0$ and $h_k^* > 0$, equation EM3 implies

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$$\text{EM5} \quad \frac{\frac{\partial u(h^*)}{\partial h_l}}{\frac{\partial u(h^*)}{\partial h_k}} = \frac{p_l}{p_k},$$

which says the marginal rate of substitution equals the ratio of prices, the same condition implied by the *UMP* problem. It is important to note however that $h(p, u) \neq x(p, w)$ in general. Therefore, while we referred to $x(p, w)$ as the Marshallian Demand, we will refer to $h(p, u)$ as the Hicksian Demand. Previously, we said that the way you distinguish the Marshallian Demand from other demands is to recognize that it depends on prices and wealth. The way to distinguish the Hicksian Demand from other notions of demand is that it depends on prices and utility.

Like Marshallian Demand, Hicksian Demand exhibits a number of interesting properties.

PROPOSITION EM1: Suppose $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation \underline{f} on $X = \mathfrak{R}_+^L$. Then for any $p \gg 0$, the Hicksian Demand $h(p, u)$ possesses the following properties:

- (i) Homogeneity of degree 0 in p : $h(ap, u) = h(p, u)$ for any p, u , and $a > 0$;
- (ii) No excess utility: For any $h \in h(p, u)$, $u(h) = u$;
- (iii) Convexity/Uniqueness: If \underline{f} is convex, then $h(p, u)$ is a convex set; and if \underline{f} is strictly convex, then $h(p, u)$ is a singleton set.

Proof: For (i), note that by the definition of a Hicksian Demand $h(ap, u) = \{h \in X: u(h) \geq u \text{ and } ap \cdot h \geq ap \cdot h' \text{ for all } h' \text{ where } u(h') \geq u \text{ and } h' \in X\} = \{h \in X: u(h) \geq u \text{ and } p \cdot h \geq p \cdot h' \text{ for all } h' \text{ where } u(h') \geq u \text{ and } h' \in X\} = h(p, u)$.

For (ii), assume that there exists an $h \in h(p, u)$ such that $u(h) > u$. Choose $a \in (0, 1)$ and define $h' = ah$. By continuity, for a close enough to 1, $u(h') = u(ah) \geq u$, but $p \cdot h > p \cdot ah = p \cdot h'$, which violates the definition of $h(p, u) = \{h \in X: u(h) \geq u \text{ and } p \cdot h \geq p \cdot h' \text{ for all } h' \text{ where } u(h') \geq u \text{ and } h' \in X\}$.

The proof of (iii) is similar to the proof of PROPOSITION CP6, so it is omitted here. Q.E.D.

Specifically, Property (i) says that increasing all prices by the same amount does not change the optimal mix of commodities needed to achieve a given level of utility at least cost. That is, only relative prices matter, which is similar to the homogeneity property for Marshallian Demand with the inclusion of wealth. Property (ii) says that we will always lower our expenditure until we just meet our minimally desired level of utility, which is similar to the satisfaction of Walras Law for Marshallian Demand. Property (iii) is analogous to the convexity and uniqueness of Marshallian Demand.

If the Hicksian Demand is unique for all $p \gg 0$, then it will also satisfy another interesting property.

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PROPOSITION EM2: Suppose $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation \underline{f} on $X = \mathfrak{R}^L_+$ and that $h(p, u)$ is a singleton set for all $p \gg 0$. Then the Hicksian Demand $h(p, u)$ satisfies the *Compensated Law of Demand*:

$$(p'' - p') \cdot (h(p'', u) - h(p', u)) \leq 0 \text{ for all } p' \text{ and } p''.$$

Proof: Since $h(p', u)$ minimizes $p' \cdot h$ by definition, $p' \cdot h(p', u) \leq p' \cdot h(p'', u)$. Since $h(p'', u)$ minimizes $p'' \cdot h$ by definition, $p'' \cdot h(p'', u) \leq p'' \cdot h(p', u)$. Adding yields $p' \cdot h(p', u) - p'' \cdot h(p'', u) \leq p'' \cdot h(p', u) - p'' \cdot h(p'', u)$, or $(p'' - p') \cdot (h(p'', u) - h(p', u)) \leq 0$. Q.E.D.

PROPOSITION EM2 is particularly important because it says that own price effects for Hicksian Demand must be nonpositive. That is, if we increase the price of a single commodity, the Hicksian Demand for that commodity must not increase. A similar result does not hold for Marshallian Demand. That is, an increase in the price of a single commodity can actually increase the Marshallian Demand for that commodity.

Like Marshallian Demand, cross price effects can go either way, which leads to the notion of Hicksian complements (when the Hicksian demand for a commodity decreases with an increase in the price of another commodity) and Hicksian substitutes (when the Hicksian Demand for a commodity increase with an increase in the price of another commodity). The effect of increasing u on Hicksian Demand is also indeterminate in general.

Plugging the Marshallian Demand back into the utility function resulted in the indirect utility function, which had a number of interesting properties. A similar exercise is to multiply the Hicksian Demand by the price vector which yields the *Expenditure Function*: $e(p, u) = p \cdot h(p, u)$. The *Expenditure Function* is the minimal level of wealth needed to achieve a particular level of utility given prices. It also has some interesting properties.

PROPOSITION EM3: Suppose $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation \underline{f} on $X = \mathfrak{R}^L_+$. The *Expenditure Function* $e(p, u) = p \cdot h(p, u)$ is

- (i) homogeneous of degree one in prices;
- (ii) strictly increasing in u and nondecreasing in p_l for any l ;
- (iii) concave in p ; and
- (iv) continuous in p and u .

Again, these properties are analogous to the properties of the indirect utility function, as are the proofs.

Before moving on to an example, there are a couple more properties of the expenditure function worth talking about.

PROPOSITION EM4: Suppose $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation \underline{f} on $X = \mathfrak{R}^L_+$. For all p and u , the Hicksian demand is the derivative vector of the expenditure function:

$$h(p, u) = \Delta_p e(p, u) \text{ or } h_l(p, u) = \frac{\partial e(p, u)}{\partial p_l} \text{ for all } l = 1, \dots, L.$$

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MWG provide three separate proofs for this result. I will provide a brief sketch of one of these assuming we have an interior solution and Hicksian Demand is differentiable in prices.

The expenditure function is defined as $e(p, u) = p \cdot h(p, u)$. Differentiating with respect to p_l yields

$$\mathbf{EM6} \quad \frac{\partial e(p, u)}{\partial p_l} = h_l(p, u) + \sum_{k=1}^L p_k \frac{\partial h_k(p, u)}{\partial p_l}$$

Equation EM3 for an interior solution implies $p_k = g^* \frac{\partial u(h^*)}{\partial h_k}$, which when substituted into equation EM6 yields

$$\mathbf{EM7} \quad \frac{\partial e(p, u)}{\partial p_l} = h_l(p, u) + g^* \sum_{k=1}^L \frac{\partial u(h^*)}{\partial h_k} \frac{\partial h_k(p, u)}{\partial p_l}.$$

Now equation EM4 implies $u = u(h(p, u))$, which when we totally differentiate with respect to p_l yields

$$\mathbf{EM8} \quad 0 = \sum_{k=1}^L \frac{\partial u(h(p, u))}{\partial h_k} \frac{\partial h_k(p, u)}{\partial p_l}.$$

Substituting equation EM8 back into equation EM7, gives us the desired result:

$$\frac{\partial e(p, u)}{\partial p_l} = h_l(p, u).$$

If the expenditure function is twice continuously differentiable in p , then we can write

$$\mathbf{EM9} \quad D_p^2 e(p, u) = \begin{bmatrix} \frac{\partial^2 e(p, u)}{\partial p_1^2} & \frac{\partial^2 e(p, u)}{\partial p_1 \partial p_2} & \mathbf{L} & \frac{\partial^2 e(p, u)}{\partial p_1 \partial p_L} \\ \frac{\partial^2 e(p, u)}{\partial p_2 \partial p_1} & \frac{\partial^2 e(p, u)}{\partial p_2^2} & & \mathbf{M} \\ \mathbf{M} & & \mathbf{O} & \mathbf{M} \\ \frac{\partial^2 e(p, u)}{\partial p_L \partial p_1} & \mathbf{L} & \mathbf{L} & \frac{\partial^2 e(p, u)}{\partial p_L^2} \end{bmatrix}.$$

With this matrix, we can highlight some additional properties of the expenditure function.

PROPOSITION EM5: Suppose $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation \underline{f} on $X = \mathfrak{R}_+^L$. If $h(p, u)$ is continuously differentiable in p and u :

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- (i) $D_p h(p, u) = D_p^2 e(p, u)$,
- (ii) $D_p h(p, u)$ is negative semidefinite,
- (iii) $D_p h(p, u)$ is symmetric, and
- (iv) $D_p h(p, u)p = 0$.

The results in PROPOSITIONS EM2, EM4, and EM5 will prove useful in helping us understand why Marshallian Demand may not have negative own price effects. It will also prove helpful when considering the possibility of recovering an individual's preferences from observed demand relations.

Now let us return to our example from the consumer problem notes. Suppose that the utility function $u(h_1, h_2) = h_1^{0.75} h_2^{0.5}$ represents an individual's preferences. Furthermore, assume the prices of h_1 and h_2 are p_1 and p_2 . The problem we are interested in solving is

EM10
$$\min_{h \geq 0} p_1 h_1 + p_2 h_2 \text{ such that } h_1^{0.75} h_2^{0.5} \geq u.$$

The Lagrangian for this problem is $L = p_1 h_1 + p_2 h_2 + \gamma(u - h_1^{0.75} h_2^{0.5})$. The first-order conditions are

EM11
$$\frac{\partial L}{\partial h_1} = p_1 - \gamma * 0.75 h_1^{-0.25} h_2^{0.5} \geq 0, \quad \frac{\partial L}{\partial h_1} h_1^* = 0, \text{ and} \quad h_1^* \geq 0,$$

EM12
$$\frac{\partial L}{\partial h_2} = p_2 - \gamma * 0.5 h_1^{0.75} h_2^{-0.5} \geq 0, \quad \frac{\partial L}{\partial h_2} h_2^* = 0, \text{ and} \quad h_2^* \geq 0,$$

EM13
$$\frac{\partial L}{\partial \gamma} = u - h_1^{0.75} h_2^{0.5} \leq 0, \quad \frac{\partial L}{\partial \gamma} \gamma^* = 0, \text{ and} \quad \gamma^* \geq 0.$$

We will again focus on an interior solution which implies,

EM14
$$p_1 - \gamma^* 0.75 h_1^{-0.25} h_2^{0.5} \equiv 0$$

EM15
$$p_2 - \gamma^* 0.5 h_1^{0.75} h_2^{-0.5} \equiv 0, \text{ and}$$

EM16
$$u - h_1^{0.75} h_2^{0.5} \equiv 0$$

Rearranging and taking the ratio of equation EM14 and EM15, we can get rid of the Lagrange multiplier and solve for h_2^* :

EM17
$$\frac{\gamma^* 0.75 h_1^{-0.25} h_2^{0.5}}{\gamma^* 0.5 h_1^{0.75} h_2^{-0.5}} \equiv \frac{p_1}{p_2} \text{ or } h_2^* \equiv \frac{p_1}{p_2} \frac{2}{3} h_1^*.$$

Substituting back into equation EM16, then yields

EM18
$$u - h_1^{0.75} \left(\frac{p_1}{p_2} \frac{2}{3} h_1^* \right)^{0.5} \equiv 0 \text{ or } h_1^* \equiv \left(\frac{3}{2} \right)^{\frac{2}{5}} \left(\frac{p_2}{p_1} \right)^{\frac{2}{5}} u^{\frac{4}{5}}.$$

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Substituting back into EM17, finally results in $h_2^* \equiv \left(\frac{2}{3}\right)^{\frac{3}{5}} \left(\frac{p_1}{p_2}\right)^{\frac{3}{5}} u^{\frac{4}{5}}$. Recall that the solution to the utility maximization problem was $x_1^* = \frac{3w}{5p_1}$ and $x_2^* = \frac{2w}{5p_2}$, which reiterates the point that Marshallian and Hicksian Demand are distinct.

The expenditure function is $e(p_1, p_2, u) = p_1 \left(\frac{3}{2}\right)^{\frac{2}{5}} \left(\frac{p_2}{p_1}\right)^{\frac{2}{5}} u^{\frac{4}{5}} + p_2 \left(\frac{2}{3}\right)^{\frac{3}{5}} \left(\frac{p_1}{p_2}\right)^{\frac{3}{5}} u^{\frac{4}{5}} =$

$\left(\left(\frac{3}{2}\right)^{\frac{2}{5}} + \left(\frac{2}{3}\right)^{\frac{3}{5}}\right) p_1^{\frac{3}{5}} p_2^{\frac{2}{5}} u^{\frac{4}{5}}$. It is pretty straightforward from here to demonstrate that this

expenditure function is homogenous of degree 1 in prices; increasing in u and nondecreasing in p_1 and p_2 ; concave in prices; and continuous in prices and u . It is also straightforward to show that differentiating $e(p_1, p_2, u)$ with respect to p_1 and p_2 yields our Hicksian Demands.