EXCHANGE ECONOMY

ECON 8001-2 **Instructor: Terry Hurley**

INTRODUCTION (MWG Chapter 10 pages 311 to 334 and 15 pages 515 to 525 & Varian Chapter 17)

We have now carefully constructed a third of the typical market economy: Aggregate Demand. The other two-thirds of the typical market economy are Aggregate Supply and Exchange. Usually, Aggregate Supply is the result of the production of commodities. Exchange reflects the interaction between Aggregate Demand and Aggregate Supply within some market institution. Before digging deeper into Aggregate Supply however, we can learn more about Aggregate Demand and Exchange by focusing in on a simple exchange economy. That is, an economy where there is no production. In such an economy, individuals are blessed with some commodity endowments and a market permits them to exchange what they have been blessed with in an effort to improve themselves. Several things that will be of interest to us are price determination, the influence of exogenous factors (such as wealth) on prices, and the extent to which markets can improve individual wellbeing. There are two ways to try to answer these questions. One way is referred to as partial equilibrium analysis. The other is referred to as general equilibrium analysis. Partial equilibrium analysis is typically less demanding than general equilibrium analysis. However, the results of a partial equilibrium analysis will not always tell us the whole story, while the results of a general equilibrium analysis will.

THE SETUP

I:	Number of individuals in our economy.
$x_i^e \in \Re^L_+$:	Vector of commodity endowments for individual i.
$x_{li}^e \geq 0$:	Specific quantity of commodity l (l = 1, 2,…, L) endowment for individual i.
$x_i \in \Re^L_+$:	Vector of commodities consumed by individual i.
$x_{li} \geq 0$:	Specific quantity of commodity l (l = 1, 2,…, L) consumed by individual i.
$\succsim_i$:	Strongly monotone, strictly convex, and continuous preference relation on $\Re^L_+$ for individual i.
$u_i: \Re^L_+ \rightarrow \Re$:	Utility function generated by $\succsim_i$.
$p^D \in \Re^L$:	Vector of prices paid for commodities by buyers.
$p_l^D \in \Re$:	Specific price paid for commodity l (for l = 1, 2,…, L) by buyers.
$P^S \in \Re^L$:	Vector of prices received for commodities by sellers.
$p_l^S \in \Re$:	Specific price received for commodity l (for l = 1, 2,…, L) by sellers.
$w_i(p^S, x_i^e) \geq 0$:	Wealth of ith individual: $w_i(p^S, x_i^e) = p^S \cdot x_i^e$.

This set up is very close to what we had for the consumer problem and Aggregate Demand. Where it differs is that individual wealth is not independent of prices. Still, the consumer problem we did before took prices as given, which in essence fixes wealth. So our solution to the consumer's utility maximization problem is still $x_i(p^D, w_i(p^S, x_i^e))$ with indirect utility function $v_i(p^D, w_i(p^S, x_i^e)) = u_i(x_i(p^D, w_i(p^S, x_i^e)))$. Analogous to before, we can construct aggregate demand by summing: $x^D = x(p^D, p^S \cdot x_1^e, \ldots, p^S \cdot x_I^e) = \sum_{i=1}^{I} x_i\left(p^D, w_i\left(p^S, x_i^e\right)\right)$, which gives us a third of the market. It is important to note that this aggregate demand is homogeneous of degree 0 in prices.

To get the other two-thirds of the market, we need to know something about Aggregate Supply and something about the market institution. For Aggregate Supply, since there is no production, what is available to the market is the total endowment of commodities: $x^S = \sum_{i=1}^{I} x_i^e$. For the market institution, we will begin our analysis by assuming there is no excess supply (e.g. $x^S > x^D$) and no excess demand (e.g. $x^D > x^S$), which implies the market clearing condition $x^S = x^D$. We will also assume consumers are price takers and that the price buyers pay is the same as the price sellers receive: $p^S = p^D = p$.

PARTIAL EQUILIBRIUM ANALYSIS

In a partial equilibrium analysis, we focus on the markets for a subset of the commodities actually traded; often a single commodity. To review this type of analysis, we will focus on commodity k. Bringing together our three pieces, we have:

EE1 $x_k^D = x_k(p^D, p^S \cdot x_1^e, \ldots, p^S \cdot x_I^e)$,

EE2 $x_k^S = \sum_{i=1}^{I} x_{ki}^e$,

EE3 $x_k^S = x_k^D$, and

EE4 $p_k^S = p_k^D = p_k$, or

EE5 $x_k(p_k, p_{\sim k}^D, p_k x_{k1}^e + p_{\sim k}^S \cdot x_{\sim k1}^e, \ldots, p_k x_{kI}^e + p_{\sim k}^S \cdot x_{\sim kI}^e) = \sum_{i=1}^{I} x_{ki}^e$

where $p_{\sim k}^D = (p_1^D, \ldots, p_{k-1}^D, p_{k+1}^D, \ldots, p_L^D)$, $p_{\sim k}^S = (p_1^S, \ldots, p_{k-1}^S, p_{k+1}^S, \ldots, p_L^S)$, and $x_{\sim k}^e = (x_k^e, \ldots, x_{k-1}^e, x_{k+1}^e, \ldots, x_L^e)$. While we are assuming buyers and sellers face the same price for the good of interest, we are allowing these prices to differ for other commodities to make our arguments more general. All this may seem like overkill, but it clearly lays out the details of what it means to conduct a partial equilibrium analysis. Note that we need not have market clearing (i.e. supply equal to demand) in any market other than the one we are exploring. Prices paid and received in other markets need not be equal in this partial equilibrium analysis.

Assuming a solution to equation EE5 exists and is unique, we can write the competitive equilibrium price as $p_k^* = p_k(p_{\sim k}^D, p_{\sim k}^S, x_1^e, \ldots, x_I^e)$. That is, it depends on the prices paid and received for other goods and each consumer's initial endowment. To ascertain, how the equilibrium price depends on these factors we can either totally differentiate or use the implicit function theorem (assuming things are actually differentiable). For example, suppose the price consumers pay to purchase good l ($l \neq k$) increases. The implicit function theorem then implies

EE6
$$\frac{\partial p_k\left(p_{\sim k}^D, p_{\sim k}^S, x_1^e,\ldots,x_I^e\right)}{\partial p_l^D} = -\frac{\dfrac{\partial x_k}{\partial p_l^D}}{\dfrac{\partial x_k}{\partial p_k}}$$

where functional arguments are suppressed to ease exposition. Note that

EE7
$$\frac{\partial x_k}{\partial p_l^D} = \sum_{i=1}^{I}\frac{\partial x_{ki}}{\partial p_l^D} = \sum_{i=1}^{I}\left(\frac{\partial h_{ki}}{\partial p_l^D} - \frac{\partial x_{ki}}{\partial w_i}x_{li}\right), \text{ and}$$

EE8
$$\begin{aligned}\frac{\partial x_k}{\partial p_k} &= \sum_{i=1}^{I}\left(\frac{\partial x_{ki}}{\partial p_k} + \frac{\partial x_{ki}}{\partial w_i}\frac{\partial w_i}{\partial p_k}\right) \\ &= \sum_{i=1}^{I}\left(\frac{\partial h_{ki}}{\partial p_k} + \frac{\partial x_{ki}}{\partial w_i}\left(x_{ki}^e - x_{ki}\right)\right).\end{aligned}$$

Equation EE7 shows how a change in the equilibrium price is affected by individual cross-price and wealth effects, while equation EE8 shows how the result is affected by individual own-price and income effects. In general, we cannot say how things will turn out unless we are willing to place additional restrictions on individual demands.

Figure 1 illustrates the notion of market equilibrium embodied in equation EE5, while Figure 2 illustrates the notion of a change in market equilibrium due to a change in the price of commodity *l* embodied in equations EE6, EE7, and EE8. The top half of Figure 2 reflects a scenario where the net effect of a change in the price of commodity *l* is a decrease in aggregate demand that results in an equilibrium price decrease for commodity *k*, while the bottom half reflects a scenario where the net effect of a change in the price of commodity *l* is an increase in aggregate demand that results in an equilibrium price increase for commodity *k*. These illustrations should be familiar.

Now before we get too far ahead of ourselves, note that we found the equilibrium price in this market by assuming it will occur precisely where aggregate demand and aggregate supply are equal. What motivates this assumption? The argument goes something like this. If aggregate demand is greater than aggregate supply at some price, this price cannot be the equilibrium price because not all buyers will be able to purchase as much as they want. Some buyers will then be willing to offer to pay a higher price in an effort to make a purchase. This can continue up to the point where aggregate demand equals aggregate supply. Alternatively, if aggregate supply is greater than aggregate demand at a given price, this price cannot be an equilibrium price because some sellers will not be able to sell as much as they want. Some sellers will then be willing to offer to sell at a lower price in order to make a sale. Again, this can continue down to the point where aggregate demand equals aggregate supply. Now does all this really work as advertised? Yes, at least much more often than it doesn't work. Some of the more interesting evidence showing how this works comes from the experimental economics literature.

So, we have figured out equilibrium price determination in a single market and how to explore changes in the equilibrium price due to changes in exogenous factors like the price of other goods or endowments. What we want to talk about now is welfare. We have already talked about the notion of using consumer surplus to measure the welfare of consumers using individual Marshallian demands. These methods can also be applied to aggregate demand:

EE9 $$CS(p_k^*, p_{\sim k}^D, p_k^* x_{k1}^e + p_{\sim k}^S \cdot x_{\sim k1}^e, \ldots, p_k^* x_{kI}^e + p_{\sim k}^S \cdot x_{\sim kI}^e) = \int_{p_k^*}^{\infty} x_k\left(p_k, p_{\sim k}^D, p_k^* \cdot x_{k1}^e + p_{\sim k}^S \cdot x_{\sim k1}^e, \ldots, p_k^* \cdot x_{kI}^e + p_{\sim k}^S \cdot x_{\sim kI}^e\right) dp_k .$$

While we know this is not a perfect measure, we will use it without apologies because the alternative is to make very restrictive assumptions on individual preferences (see MWG). Note that in equation EE9 and EE10, we are holding $p_k^S = p_k^*$ when we do the integration. Why? The reason why is because we defined consumer surplus holding wealth constant, which is inconsistent with allowing p_k^S to change.

What we need to think about now is how do sellers benefit from trade? We will get into this in a fair amount of generality in ECON 8002, so for know I will just tell you that the typical way to measure the benefits to sellers in partial equilibrium is what is called *Producer Surplus*, which we can measure as the difference in the price a seller receives and its cost of production. Since we have no cost of production in our simple exchange economy, producer surplus is just the price the seller receives times the amount traded:

EE10 $$PS(p_k^*, x_{k1}^e, \ldots, x_{kI}^e) = p_k^* \sum_{i=1}^{I} x_{ki}^e .$$

Adding up consumer and producer surplus gives us a total measure of welfare which we will refer to as the *Total Surplus*:

EE11 $$TS(p_k^*, x_{k1}^e, \ldots, x_{kI}^e) = CS(p_k^*, p_{\sim k}^D, p_k^* x_{k1}^e + p_{\sim k}^S \cdot x_{\sim k1}^e, \ldots, p_k^* x_{kI}^e + p_{\sim k}^S \cdot x_{\sim kI}^e) + PS(p_k^*, x_{k1}^e, \ldots, x_{kI}^e).$$

Figure 3 illustrates the consumer, producer, and total surplus. An important result you should recall from previous economics courses is that the total surplus in this market will be maximized at the equilibrium price. Something to think about is what would happen if we introduce a tax or subsidy on commodity *k*? How would the total surplus change?

The general concepts we have dealt with here should all be quite familiar, though we have admittedly complicated things in order to develop a more complete understanding of what is really going on.

GENERAL EQUILIBRIUM

Partial equilibrium analysis ignores some markets, which means that a partial equilibrium analysis may not give us the whole story. *General Equilibrium* analysis rectifies this shortcoming by simultaneously considering all markets. General equilibrium analysis also dispenses with using consumer surplus (or even equivalent or compensating variation) to measure welfare and instead relies on the concept of *Pareto Efficiency*, which has stronger theoretical foundations. We will start our foray into general equilibrium analysis by considering how price is determined. We will then look at how we can evaluate changes in prices when exogenous factors change. Finally, we will consider how welfare analysis is conducted.

In our partial equilibrium analysis, we brought together the aggregate demand and supply for a commodity in a market and assumed the equilibrium price would eliminate any excess supply and excess demand. We can generalize this notion for general equilibrium:

EE1 $x^D = x(p^D, p^S \cdot x_1^e, \ldots, p^S \cdot x_I^e)$,

EE2 $x^S = \sum_{i=1}^{I} x_i^e$,

EE3 $x^S = x^D$, and

EE4 $p^S = p^D = p$,

which all imply

EE5 $x(p, p \cdot x_1^e, \ldots, p \cdot x_I^e) = \sum_{i=1}^{I} x_i^e$.

For our partial equilibrium analysis, we assumed that there was no excess supply and no excess demand in order to move the discussion forward quickly. Here we will be a bit more formal in terms of how we define equilibrium, so that we can argue for existence.

DEFINITION:
A *Walrasian Equilibrium* is a vector of prices p^* and aggregate demand $x^D = x(p, p \cdot x_1^e, \ldots, p \cdot x_I^e)$ such that $x(p^*, p^* \cdot x_1^e, \ldots, p^* \cdot x_I^e) \leq \sum_{i=1}^{I} x_i^e$.

Aggregate Excess Demand ($z(p)$): The amount by which aggregate demand exceeds aggregate supply: $z(p) = x(p, p \cdot x_1^e, \ldots, p \cdot x_I^e) - \sum_{i=1}^{I} x_i^e$.

Desirable Commodity: A commodity with positive aggregate excess demand when its prices is 0: if $p_l = 0$, then $z_l(p) > 0$.

We are now ready to prove some useful results that will lead us to the conclusion that equilibrium prices must exist.

PROPOSITION EE1: For any $p > 0$, the value of excess demand must be 0: $p \cdot z(p) \equiv 0$.

Proof: The proof of this result follows immediately from *Walras Law*. $p \cdot z(p) = p \cdot \left(x\left(p, px_1^e, \ldots, px_I^e\right) - \sum_{i=1}^{I} x_i^e \right) = \sum_{i=1}^{I} p \cdot x_i\left(p, px_i^e\right) - p \cdot x_i^e = 0$ because $p \cdot x_i(p, p \cdot x_i^e) = p \cdot x_i^e$ by Walras Law. Q.E.D.

An important corollary follows immediately from PROPOSITION EE1:

COROLLARY EE1: If demand equals supply in markets $1, \ldots, l - 1, l + 1, \ldots, L$ and $p_l > 0$, then demand must equal supply in the lth market.

Simply put COROLLARY EE1 says that the L markets are not independent.

PROPOSITION EE2: If p^* and $x^D = x(p, p \cdot x_1^e, \ldots, p \cdot x_I^e)$ are a Walrasian Equilibrium and $z_l(p) < 0$, then $p_l = 0$.

Proof: Since p^* and $x^D = x(p, p \cdot x_1^e, \ldots, p \cdot x_I^e)$ are a Walrasian Equilibrium, $x(p^*, p^* \cdot x_1^e, \ldots, p^* \cdot x_I^e) \leq \sum_{i=1}^{I} x_i^e$, which implies $p_l^* x_l(p^*, p^* \cdot x_1^e, \ldots, p^* \cdot x_I^e) \leq \sum_{i=1}^{I} p_l^* x_{li}^e$. Now if $z_l(p) < 0$ and $p_l > 0$, then $p_l z_l(p) < 0$, so summing implies $p^* \cdot x(p^*, p^* \cdot x_1^e, \ldots, p^* \cdot x_I^e) < \sum_{i=1}^{I} p^* \cdot x_i^e$ or $p^* \cdot z(p^*) < 0$. But this violates PROPOSITION EE1. Q.E.D.

PROPOSITION EE2 says that in a Walrasian Equilibrium the aggregate supply of a commodity must equal the aggregate demand for it unless the price of the commodity is 0 (i.e. the commodity is free).

PROPOSITION EE3: If all goods are desirable and p^* and $x^D = x(p, p \cdot x_1^e, \ldots, p \cdot x_I^e)$ is a Walrasian Equilibrium, then $z(p^*) = 0$.

Proof: Suppose not such that there is some commodity l where $z_l(p^*) < 0$. Then by PROPOSITION EE2, $p_l^* = 0$. But this contradicts the definition of a desirable commodity for l: if $p_l = 0$, then $z_l(p) > 0$. Q.E.D.

DEFINITION:

$$S^{L-1} = \left\{ s \in \Re_+^{\,L} : \sum_{l=1}^{L} s_l = 1 \right\}$$

PROPOSITION EE4: If z: $S^{L\text{-}1} \rightarrow \hat{A}^L$ is a continuous function that satisfies $p \cdot z(p) \equiv 0$, then there exists some $p^* \in S^{L\text{-}1}$ such that $z(p^*) \leq 0$.

We will not prove PROPOSITION EE4 because it is fairly technical and requires the use of fixed point theorems, which are not typically covered in the prerequisites for this course.

PROPOSITION EE4 is essentially saying that a Walrasian Equilibrium must exist provided our excess demand functions are continuous and Walras Law holds. There is however an important caveat that we need to recognize. Notice that PROPOSITION EE4 says $p^* \in S^{L\text{-}1}$, which means that this price vector has $L - 1$ elements. But we said that there are L different commodities not $L - 1$. What is going on here? Recall that COROLLARY EE1 implied that our L markets are not independent, so if we know what the equilibrium is in $L - 1$ markets, we automatically know what the equilibrium is in the final market. All this follows from the fact that demand is homogenous of degree 0 in prices and wealth, which we said implies only relative prices matter. Therefore, to solve for a Walrasian Equilibrium, we must normalize prices. One way to accomplish this normalization is to set the price of one commodity equal to 1. Another alternative used in the proof of PROPOSITION EE4 is to normalize such that $\sum_{l=1}^{L} p_l = 1$ and there are only $L - 1$ independent prices.

A common device used to show how price determination works in general equilibrium is the *Edgeworth Box*. To illustrate, we will assume that there are only two consumers (labeled A and B) and two goods (labeled 1 and 2). The dimensions of the Edgeworth Box are the total endowment for each good: $x_1^e = x_{1A}^e + x_{1B}^e$ and $x_2^e = x_{2A}^e + x_{2B}^e$ (see Figure 4). In Figure 4, the consumption of commodities by Consumer A is measured in the Northeast direction, while the consumption of commodities by Consumer B is measured in the Southwest direction. The initial endowments are x_{1A}^e and x_{2A}^e for Consumer A and x_{1B}^e and x_{2B}^e for consumer B. Notice that these are indicated by the same point because the total dimension of the box is the sum of these endowments.

Within this Edgeworth Box, we can use our standard consumer theory to define utility functions and indifference curves. Figure 5 illustrates four representative indifference curves for Consumer A. Note that with these indifference curves $u_A^3 > u_A^2 > u_A^1 > u_A^0$ because consumption is increasing to the Northeast. Figure 6 illustrates four representative indifference curves for Consumer B. For these indifference curves, $u_B^3 > u_B^2 > u_B^1 > u_B^0$ because consumption is increasing to the Southwest.

Given a consumer's initial endowment, $x_i(p, p{\cdot}x_1^e, \ldots, p{\cdot}x_I^e)$ characterizes what is referred to as an *Offer Curve*. An *Offer Curve* illustrates how an individual's demand for each commodity changes as prices change, while holding endowments fixed. Figure 7 illustrates for Consumer A. We can start by finding the indifference curve that runs through consumer A's endowment: labeled u_A^2 in the figure. We can then find a budget constraint that is just tangent to this indifference curve at the endowment: labeled p^0 in the figure. Now, we change the relative prices to trace out new budget constraints that run through the initial endowment: p^1 and p^2 in the figure. Finally, we can identify the consumer's optimal commodity bundles given these alternative budget constraints by looking for tangencies with ever higher indifference curves: u_A^5

and u_A^6. The commodity bundles on this offer curve are affordable for the consumer and maximize utility given the proposed prices. Another important point to note is that all points on the offer curve must be on or above the indifference curve running through the commodity endowment, otherwise the consumer would not be willing to trade any of his endowment.

Note that we can do the same exercise for all consumers, but the question that may arise is why? To understand why, consider Figure 8 where I have sketched offer curves for Consumer *A* and *B*. Notice that these offer curves intersect at x_{1A}^* and x_{2A}^*, and x_{1B}^* and x_{2B}^* inside the region where Consumer *A*'s and *B*'s indifference curves overlap. Note that if they are ever going to intersect, it has to be in this region because the offer curves must lie on or above u_A^2 and u_B^1. Also notice the budget constraint tangent to u_A^2 at the endowment (labeled p^0); tangent to u_B^1 at the endowment (labeled p^4); and passing through the endowment and the intersection of these two offers curves. What do these budget constraints tell us? For p^0, x_{1A}^e and x_{2A}^e is optimal for Consumer *A*, while x_{1B}^0 and x_{2B}^0 is optimal for Consumer *B*. But $x_{1A}^e + x_{1B}^0 > x_1^e$ and $x_{2A}^e + x_{2B}^0 < x_2^e$, which means there is excess demand for commodity 1 and excess supply for commodity 2, which cannot be a Walrasian Equilibrium. For p^4, x_{1A}^4 and x_{2A}^4 is optimal for Consumer *A*, while x_{1B}^e and x_{2B}^e is optimal for Consumer *B*. But $x_{1A}^4 + x_{1B}^e < x_1^e$ and $x_{2A}^4 + x_{2B}^e > x_2^e$, which means there is excess supply for commodity 1 and excess demand for commodity 2, which cannot be a Walrasian Equilibrium. For p^*, x_{1A}^* and x_{2A}^* is optimal for Consumer *A*, while x_{1B}^* and x_{2B}^* is optimal for Consumer *B*. Also, $x_{1A}^* + x_{1B}^* = x_1^e$ and $x_{2A}^* + x_{2B}^* = x_2^e$, which means there is no excess supply and no excess demand for either commodity, which can be a Walrasian Equilibrium. Therefore, any point where the offer curves intersect will be a Walrasian Equilibrium.

With a better understanding of price determination in general equilibrium, it is now time to consider the relationship between exogenous factors and the equilibrium price. To do this, the first question we need to address is what is exogenous? In our simple model, the only things that are exogenous are endowments and preferences. We will focus on a change in endowments. From equation EE5 we know $x(p, p\cdot x_1^e, \ldots, p\cdot x_I^e) = \sum_{i=1}^{I} x_i^e$, but in a Walrasian Equilibrium these demands will not all be independent and we need to normalize price. Let us set $p_1^* = 1$ and totally differentiate the first $L - 1$ market clearing demand equations assuming our equilibrium price vector is unique and everything is differentiable:

EE6 $$\sum_{k=2}^{L}\left(\frac{\partial x_l}{\partial p_k} + \sum_{i=1}^{I}\frac{\partial x_l}{\partial w_i}x_{ki}^e\right)dp_k^* = \sum_{k=2}^{L}\sum_{i=1}^{I}\left(1 - \frac{\partial x_l}{\partial w_i}p_k^*\right)dx_{ki}^e \quad \text{for } l = 1,\ldots, L-1.$$

Equation EE6 can be written in matrix form as $\Omega dp^* = \Phi dx^e$ where Ω has dimensions $L-1 \times L-1$, dp^* has dimensions $L-1 \times 1$, Φ has dimensions $L-1 \times I(L-1)$, and dx^e has dimensions $I(L-1) \times 1$. This system of equations can be solved using tools like Cramer's Rule. There is not much insightful to be gained by tackling this problem in generality (even if we just do a 2×2 Edgeworth box), so we will instead move on to Welfare analysis.

With partial equilibrium analysis, we talked about welfare using consumer surplus. As mentioned earlier, consumer surplus has its problems in terms of trying to relate it directly back to individual preferences and welfare. In general equilibrium analysis, we will use the concept of *Pareto Efficiency* in order to develop more solid theoretical underpinnings for our welfare analysis.

DEFINITION:
An economic outcome is *Pareto Efficient* if there is no other feasible economic outcome where every individual is at least as well off and some individual is strictly better off.

In terms of the economy we have been talking about, $(x_1^{PO},\ldots, x_I^{PO})$ where $\sum_{i=1}^{I} x_i^{PO} = \sum_{i=1}^{I} x_i^{e}$ is Pareto Efficient if there is no $(x_1,\ldots, x_I)$ such that $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} x_i^{e}$ and $u_i(x_i) > u_i(x_i^{PO})$ for some $i \in \{1,\ldots, I\}$.

There is a relatively straightforward way to identify Pareto Efficient allocations of commodities. We can solve the problem

EE6 $$\max_{x_1\geq 0,\ldots,x_I\geq 0} u_1(x_1) \text{ subject to } u_i(x_i) \geq u_i \text{ for } i = 2,\ldots,I \text{ and}$$
$$\sum_{i=1}^{I} x_{li}^{e} \geq \sum_{i=1}^{I} x_{li} \text{ for } k = 1,\ldots, L.$$

The Lagrangian for this problem can be written as

EE7 $$L = u_1(x_1) + \sum_{i=2}^{I} l_i\left(u_i(x_i) - u_i\right) + \sum_{l=1}^{L} g_l\left(\sum_{i=1}^{I}\left(x_{li}^{e} - x_{li}\right)\right)$$

which yields the first order conditions

EE8 $$\frac{\partial L}{\partial x_{k1}} = \frac{\partial u_1\left(x_1^*\right)}{\partial x_{k1}} - g_k^* \leq 0,\ \frac{\partial L}{\partial x_{k1}} x_{k1}^* = 0, \text{ and } x_{k1}^* \geq 0 \text{ for } k = 1,\ldots, L,$$

EE9 $$\frac{\partial L}{\partial x_{kj}} = l_j^* \frac{\partial u_j\left(x_j^*\right)}{\partial x_{kj}} - g_k^* \leq 0,\ \frac{\partial L}{\partial x_{kj}} x_{kj}^* = 0, \text{ and } x_{kj}^* \geq 0 \text{ for } j = 2,\ldots, I \text{ and } k = 1,\ldots, L,$$

EE10 $$\frac{\partial L}{\partial l_j} = u_j\left(x_j^*\right) - u_j \geq 0,\ \frac{\partial L}{\partial l_j} l_j^* = 0, \text{ and } l_j^* \geq 0 \text{ for } j = 2,\ldots, I, \text{ and}$$

EE11 $$\frac{\partial L}{\partial g_k} = \sum_{i=1}^{I}\left(x_{li}^{e} - x_{li}^*\right) \geq 0,\ \frac{\partial L}{\partial g_k} g_k^* = 0, \text{ and } g_k^* \geq 0 \text{ for } k = 1,\ldots, L.$$

The solution to this problem should one exist will be an allocation of commodities that can be written as $\left(x_1\left(u_2,\ldots,u_I,\sum_{i=1}^{I}x_i^e\right),\ldots,x_I\left(u_2,\ldots,u_I,\sum_{i=1}^{I}x_i^e\right)\right)$. Note that by changing u_2 through u_I, we can find other Pareto Efficient allocations; typically, there are lots of them. More importantly, if we assume an interior solution exists, the first order conditions provide insight into precisely how we should allocate our commodities. For example, equation EE8 and EE9 would imply

EE8' $\dfrac{\partial u_1(x_1^*)}{\partial x_{k1}} = g_k^*$ and

EE9' $\dfrac{\partial u_j(x_j^*)}{\partial x_{kj}} = \dfrac{g_k^*}{l_j^*}$.

Combining equation EE8' and EE9' then yields

EE13 $\dfrac{\dfrac{\partial u_j(x_j^*)}{\partial x_{lj}}}{\dfrac{\partial u_j(x_j^*)}{\partial x_{kj}}} = \dfrac{\dfrac{\partial u_1(x_1^*)}{\partial x_{l1}}}{\dfrac{\partial u_1(x_1^*)}{\partial x_{k1}}}$ for all $j = 2,\ldots, I$ and $l, k = 1,\ldots, L$,

which essentially says we should equate everyone's marginal rate of substitutions across goods.

At this point, it is worth recalling the first order conditions for the consumer's optimization problem (adding a subscript *i* where appropriate):

CP4 $\dfrac{\partial L}{\partial x_{li}} = \dfrac{\partial u_i(x_i^*)}{\partial x_{li}} - l^* p_l \le 0$, $\dfrac{\partial L}{\partial x_{li}} x_{li}^* = 0$, and $x_{li}^* \ge 0$ for $l = 1,\ldots L$, and

CP5 $\dfrac{\partial L}{\partial l} = w - p \cdot x_i^* \ge 0$, $\dfrac{\partial L}{\partial l} l^* = 0$, and $l^* \ge 0$.

If the solution to equations CP4 and CP5 is interior, then

EE14 $\dfrac{\dfrac{\partial u_i(x_i^*)}{\partial x_{li}}}{\dfrac{\partial u_i(x_i^*)}{\partial x_{ki}}} = \dfrac{p_l}{p_k}$.

But if all consumers face the same prices, equation EE14 implies exactly the same thing as equation EE13. That is, the competitive equilibrium equates the marginal rates of substitution across goods just like the solution to the Pareto Efficiency problem. Also PROPOSITION EE3 tells us that in a Walrasian equilibrium excess demand will equal 0, which implies equation E11 will be satisfied. The bottom line is that a competitive equilibrium (with all the assumptions we have imposed to date) will be Pareto Efficient:

First Fundamental Theorem of Welfare Economics: If preferences are continuous, and strongly monotonic, any Walrasian Equilibrium outcome is Pareto Efficient.

There is a followup to the first fundamental theorem of welfare economics:

Second Fundamental Theorem of Welfare Economics: If preferences are convex, continuous, and strongly monotonic, then the Pareto Efficient allocation $x_i^* >> 0$ for all $i = 1,\ldots, I$ is a Walrasian Equilibrium for endowment $x_i^e = x_i^*$ for all $i = 1,\ldots, I$.

All this can be demonstrated in our Edgeworth box. Figure 9 illustrates allocations that represent Pareto improvements relative to some initial endowments. These points are the overlap of indifference curves for *A* and *B* running through the endowments point. While all these points represent Pareto Improvements, it is important to realize that they need not all be Pareto Efficient.

Figure 10 illustrates how we can identify all possible Pareto Efficient allocations given consumers' endowments. Note that many of these points do not represent Pareto improvements given initial endowments. They are nonetheless Pareto Efficient.

Taking the intersection of these Pareto improvements and Pareto Efficient set yields what is often referred to as the *Contract Curve* (see Figure 11). This Contract Curve shows all Pareto Efficient allocations that consumers might be willing to trade to given their initial endowments.

If we let a Walrasian market frame negotiations amongst our consumers, we will end up at a particular point on this curve (assuming strict convexity), which illustrates the implication of the First Fundamental Theorem of Welfare Economics (see Figure 12).

The Second Fundamental Theorem of Welfare Economics essentially implies that if we know where we want to end up, we can get there with a Walrasian equilibrium provided we make a suitable redistribution of endowments (See Figure 13).

Before moving on to production, it is worth discussing the advantages and disadvantages of Pareto Efficiency. The normative content of Pareto Efficiency is that we can make no one better off without harming someone else. Unfortunately, giving everything to one person is Pareto Efficient provided all goods are desirable. Therefore, Pareto Efficiency does not take into account issues of equity or fairness. However, as you will see in your homework assignment. If we do think about equity and are willing to specify a social welfare function that is increasing in individual utilities taking equity into account, then any social welfare maximizing allocation will also have to be Pareto Efficient. So Pareto Efficiency will in many instances be necessary for maximizing social welfare.

Figure 1: Partial Equilibrium in an Exchange Economy

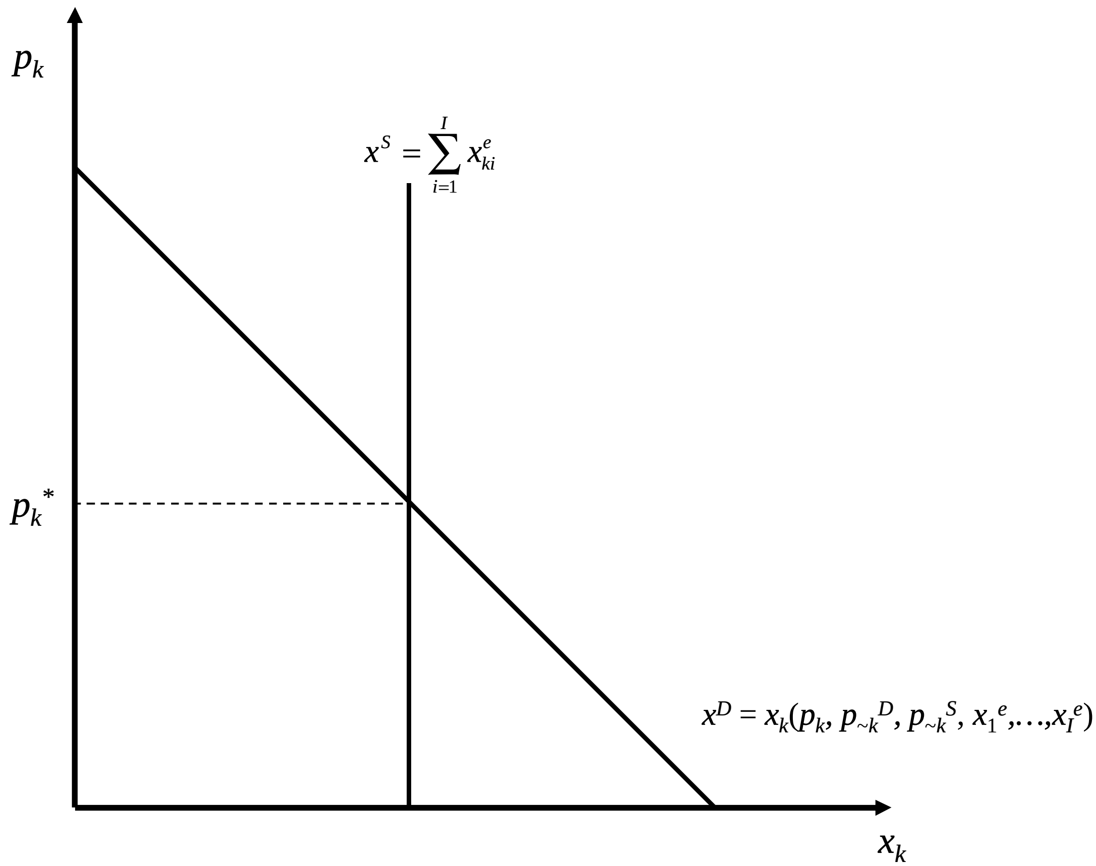


Figure 2: Demand Shifts in an Exchange Economy

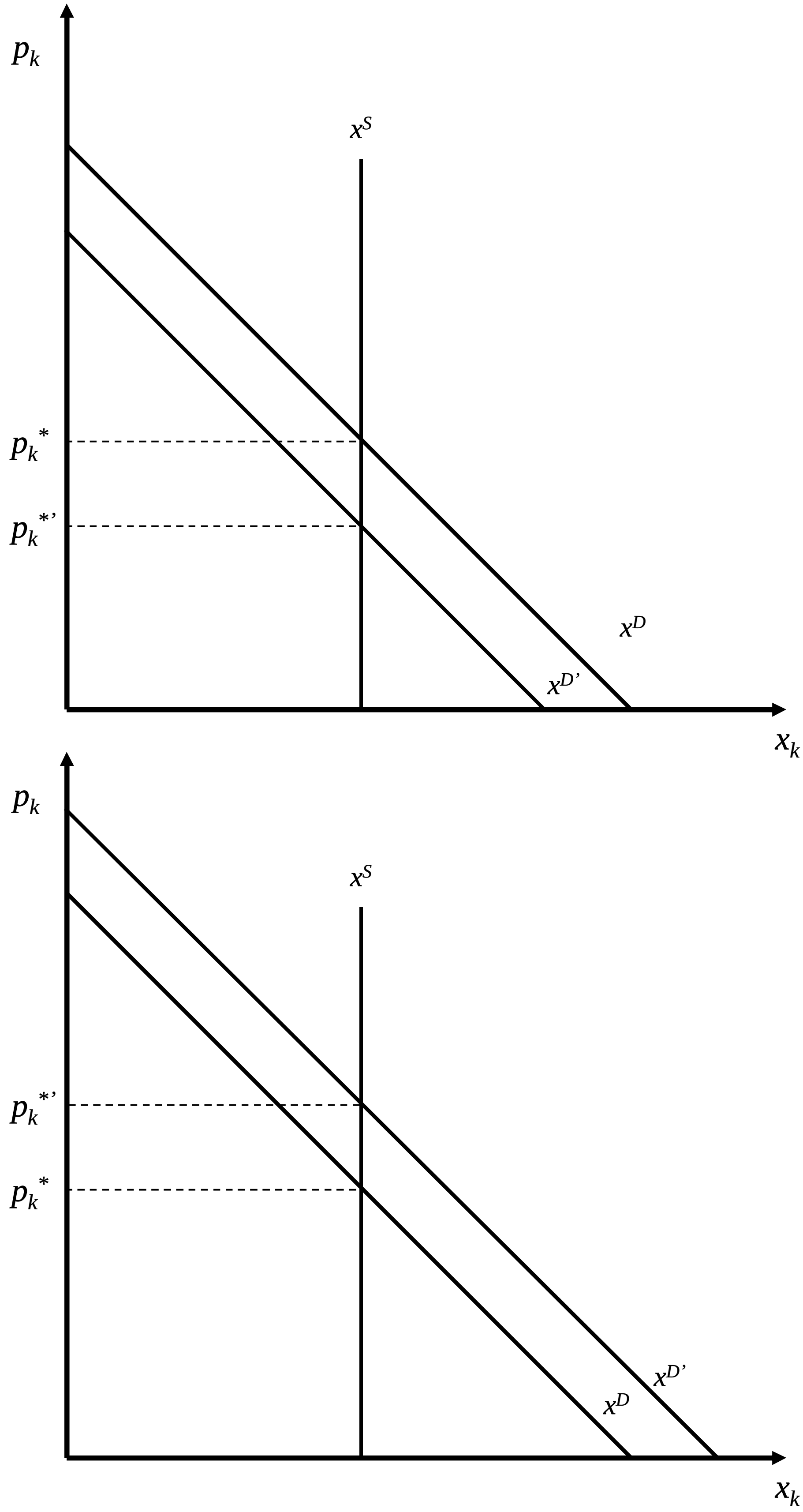
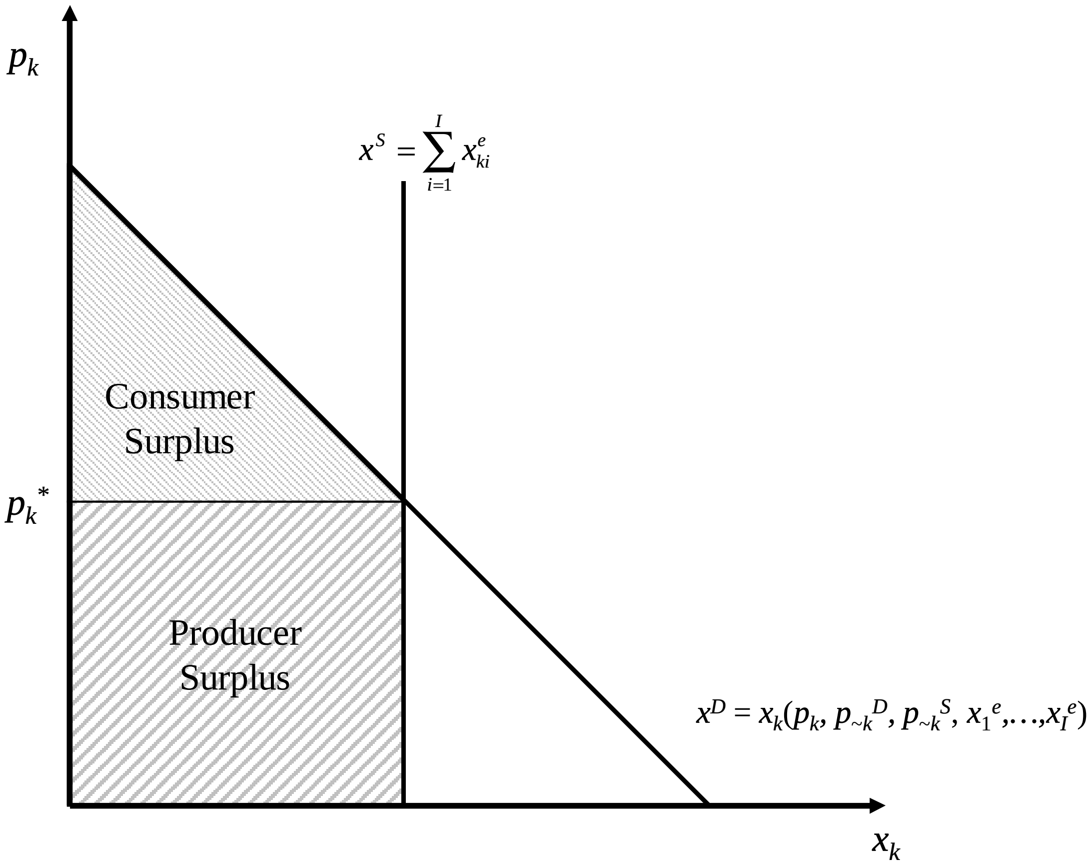


Figure 3: Consumer, Producer, & Total Surplus



Total Surplus = Consumer Surplus + Producer Surplus

Figure 4: Edgeworth Box and Initial Endowments

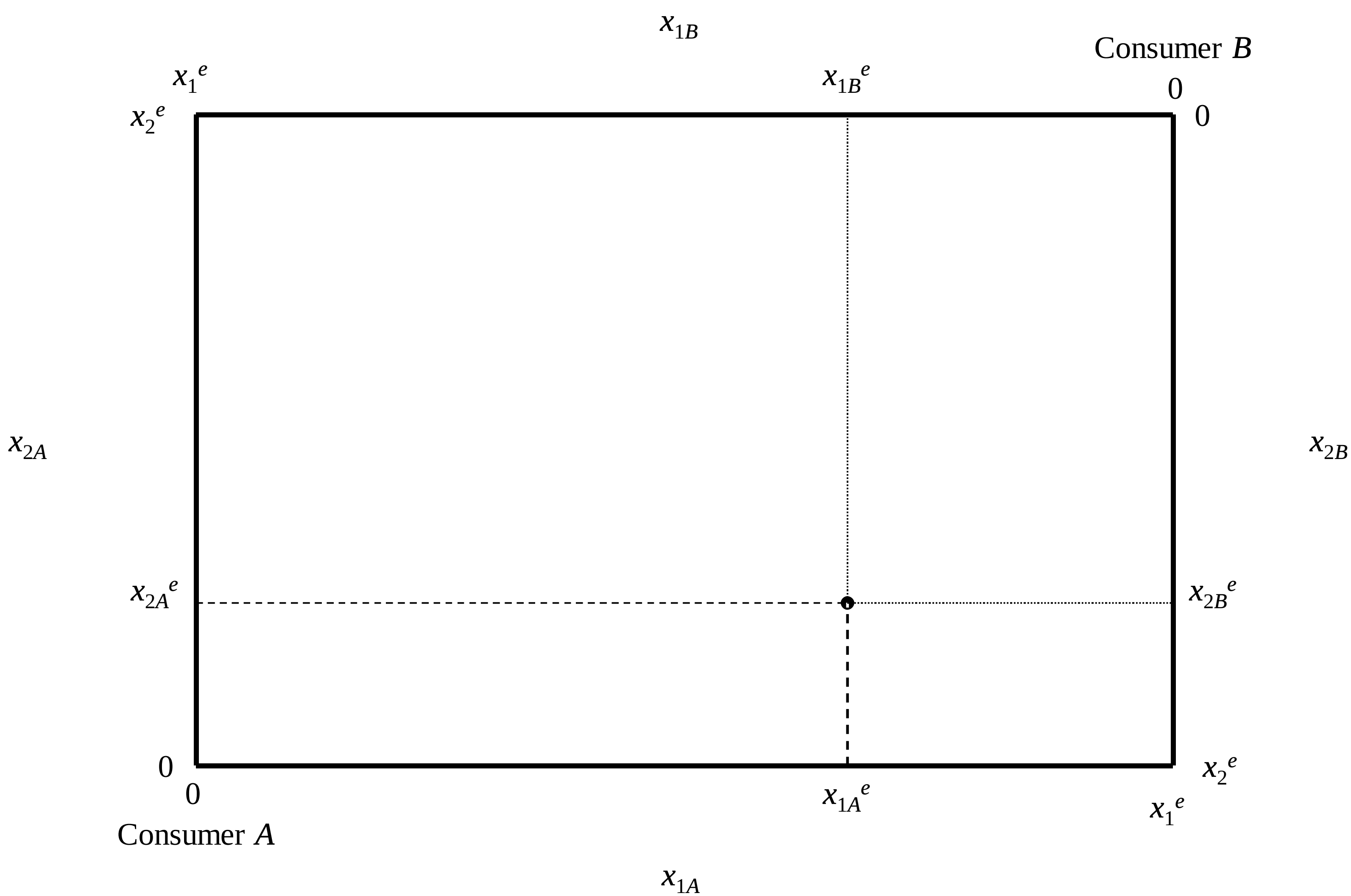


Figure 5: Edgeworth Box, Initial Endowments, and Consumer *A*'s Preferences

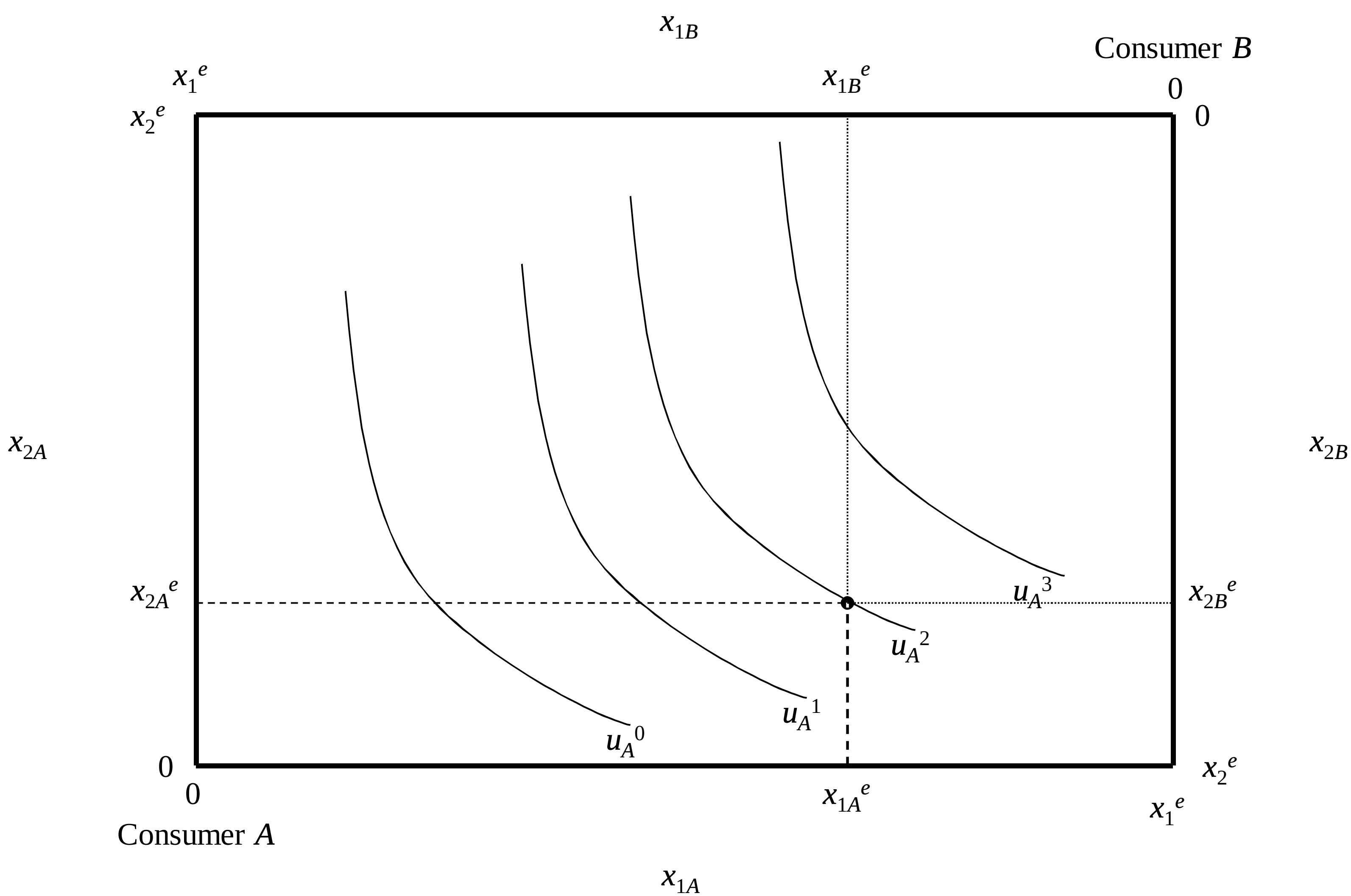


Figure 6: Edgeworth Box, Initial Endowments, and Consumer *B*'s Preferences

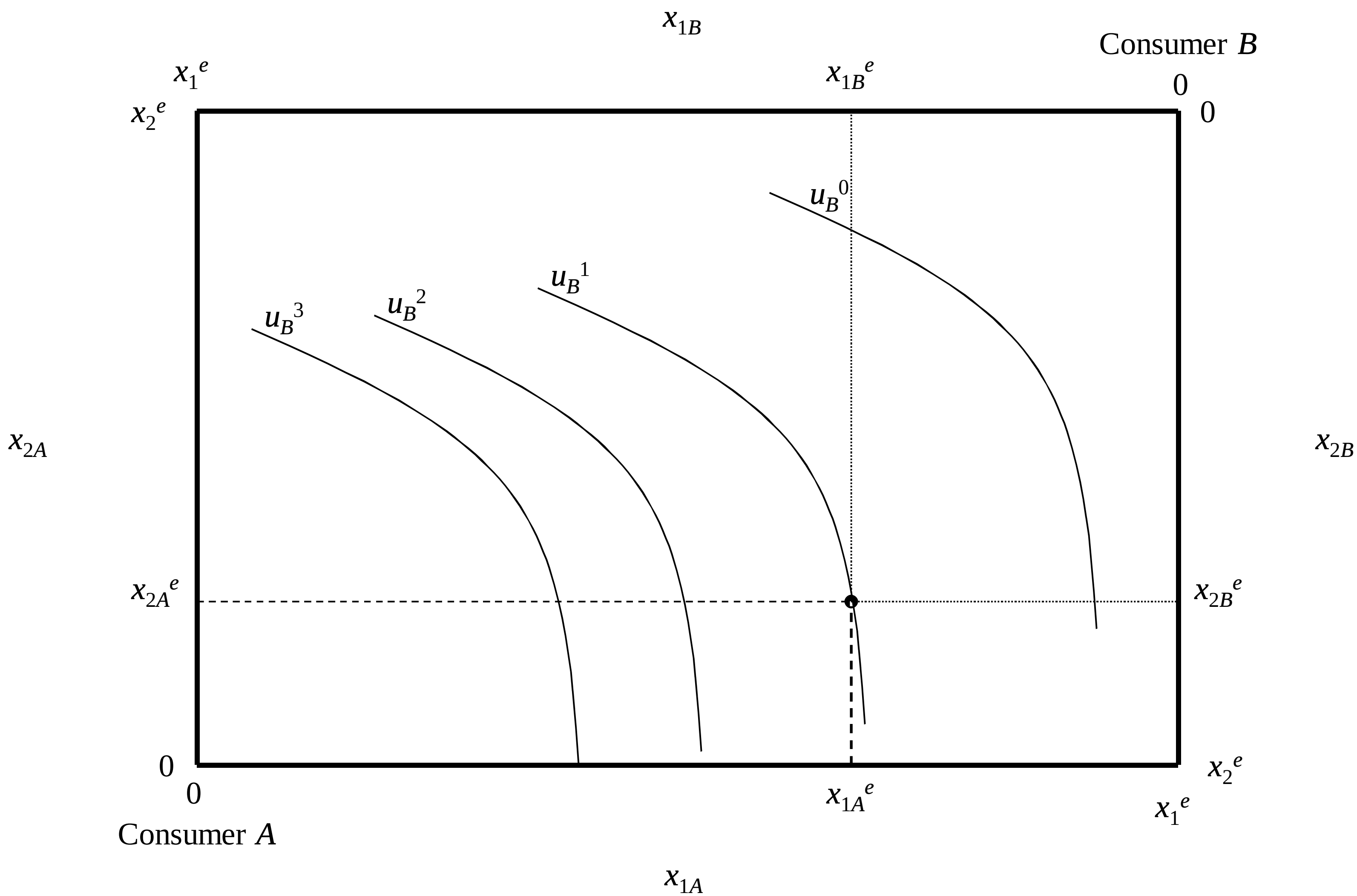


Figure 7: Edgeworth Box, Initial Endowments, and Consumer *A*'s Offer Curve

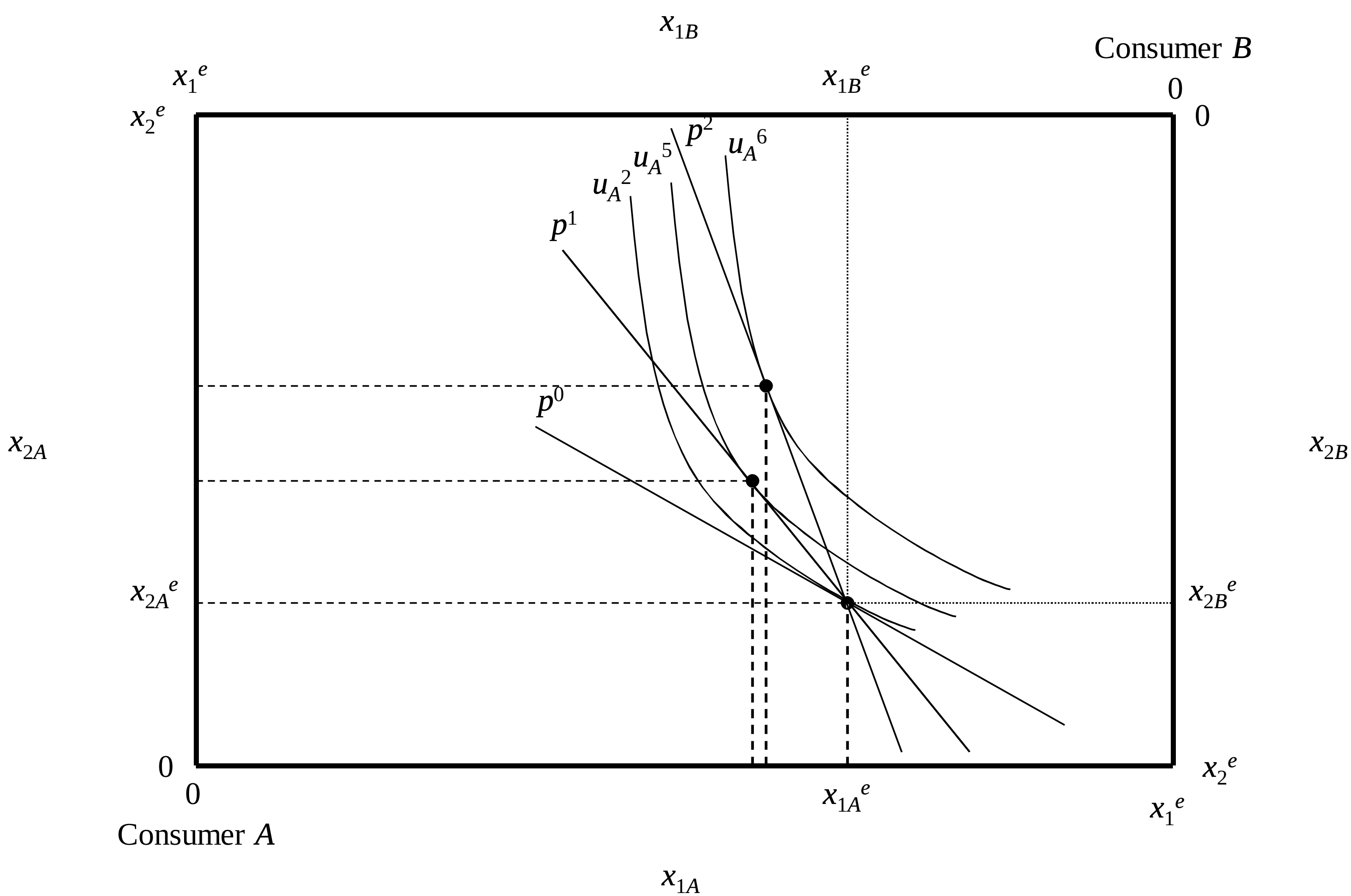


Figure 8: Edgeworth Box, Initial Endowments, Consumer *A*'s and *B*'s Offer Curves, and Walrasian Equilibrium

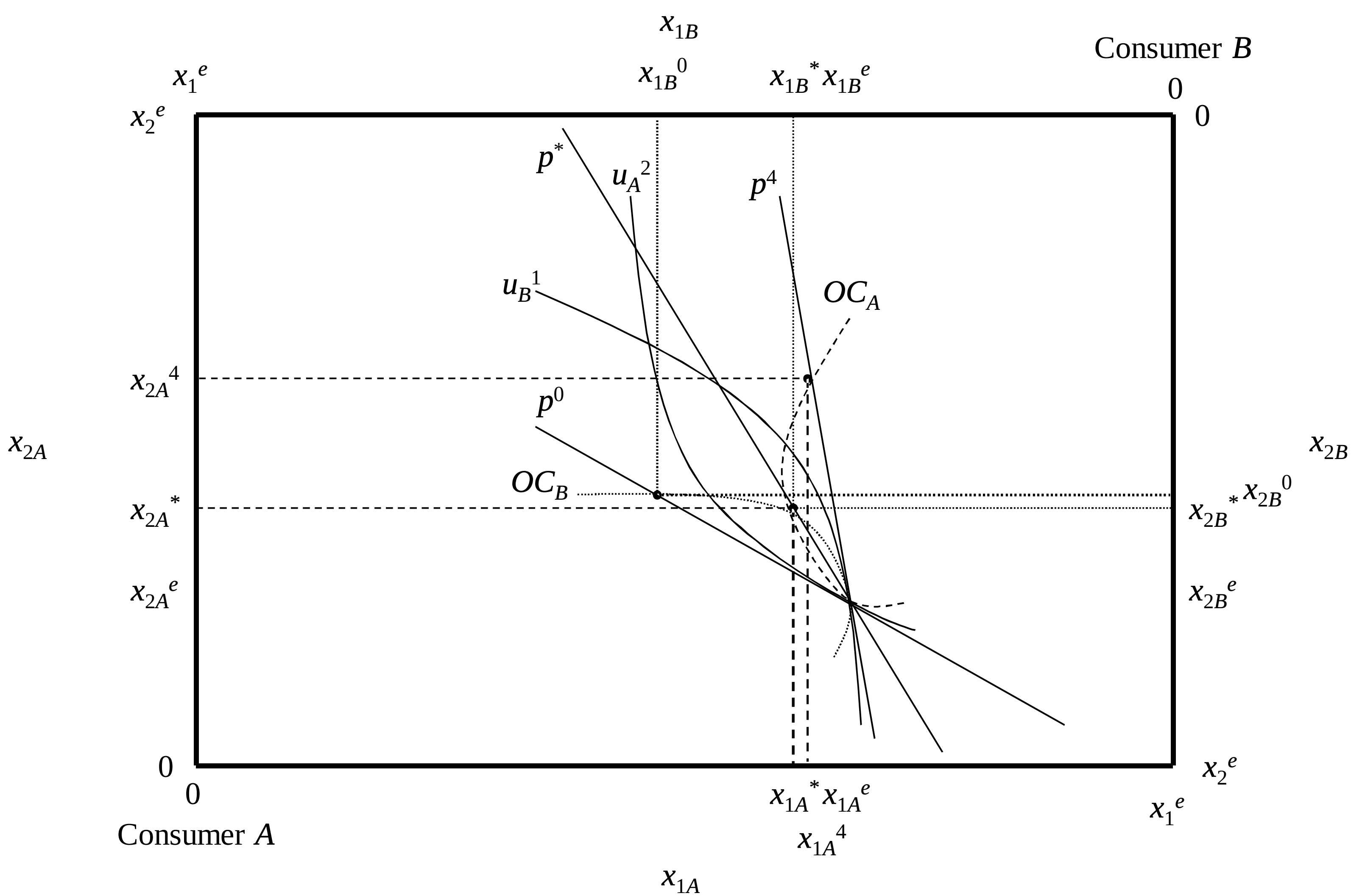


Figure 9: Pareto Improving Allocations

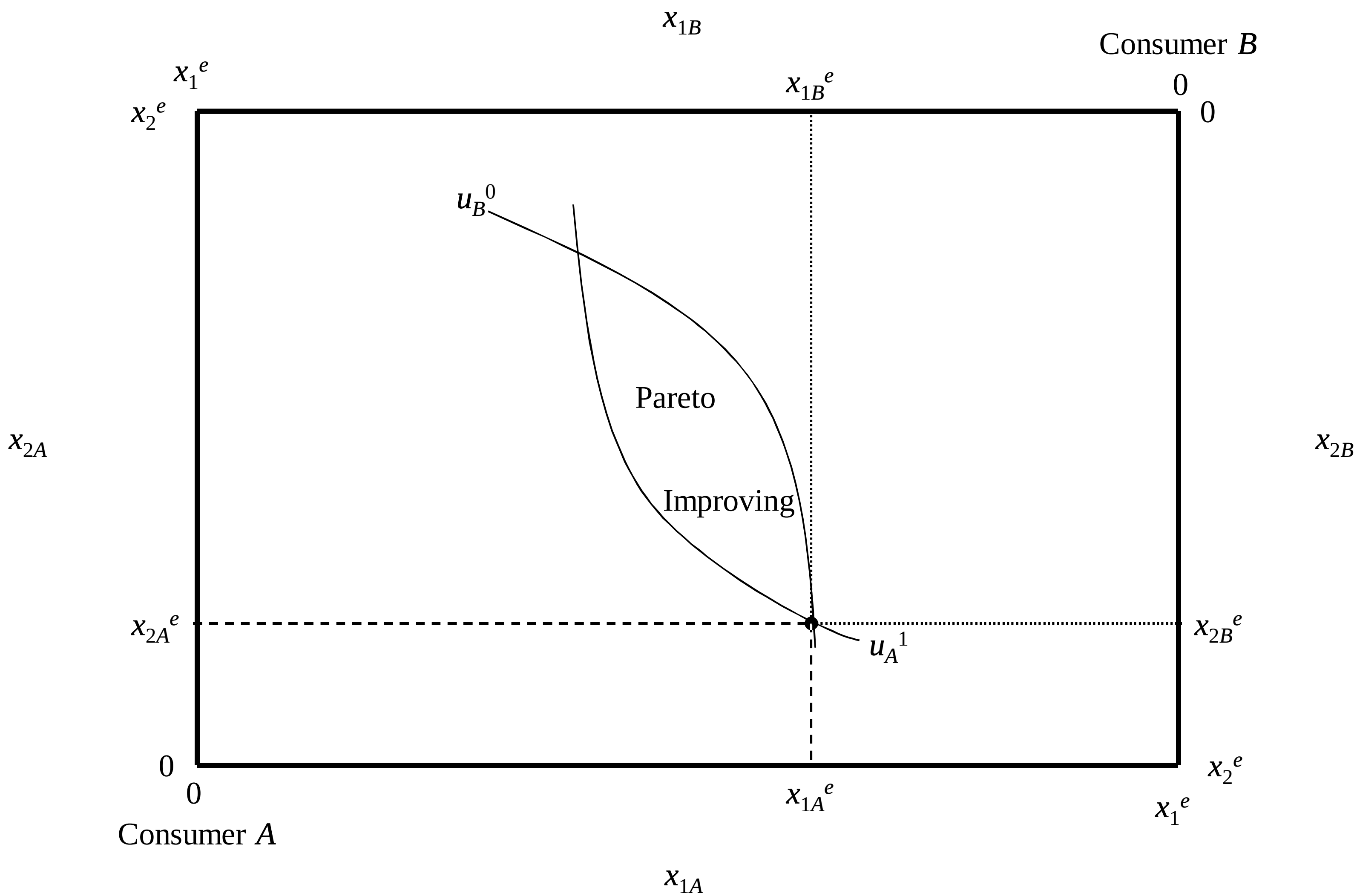


Figure 10: Pareto Efficient Allocations

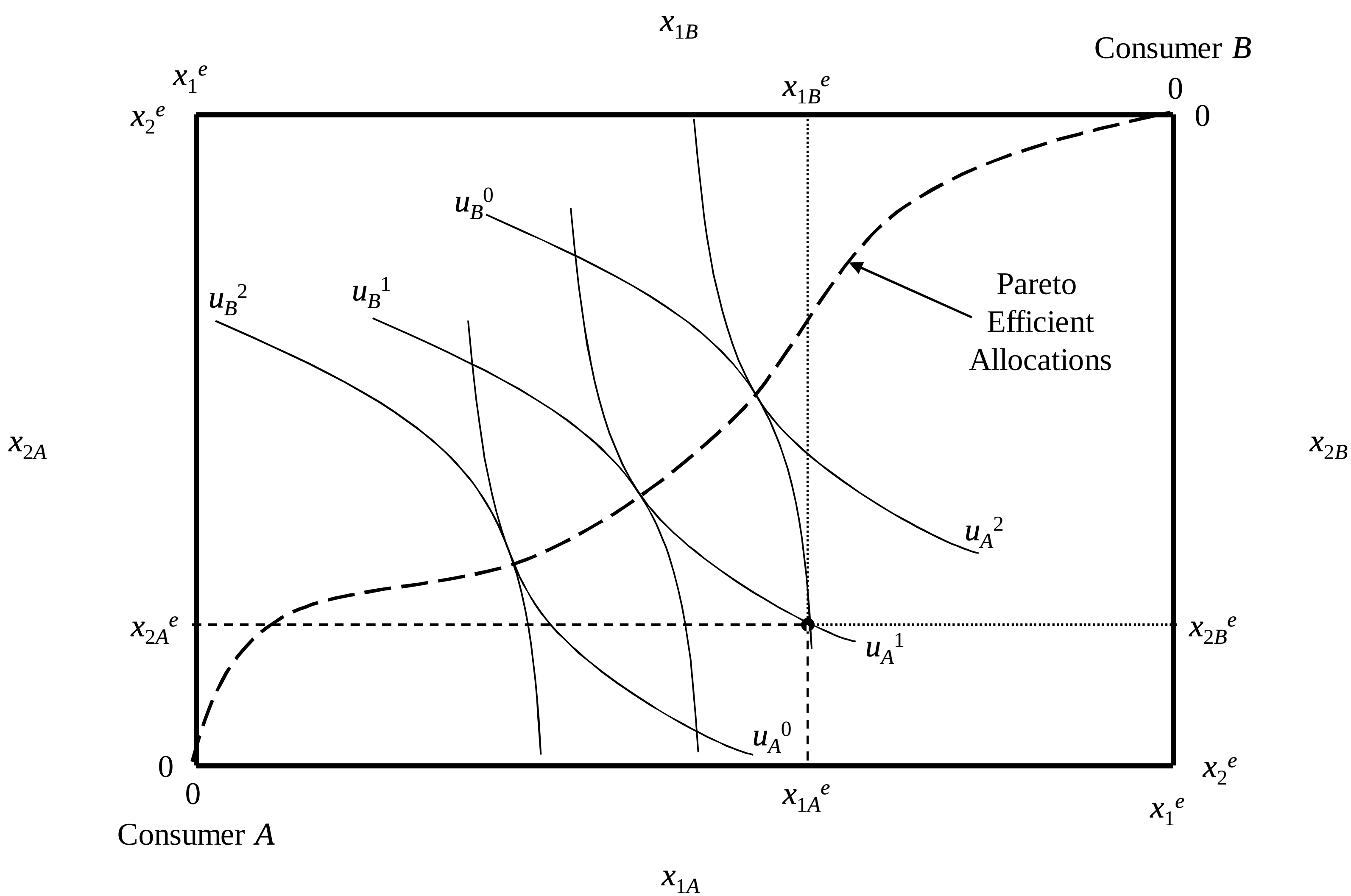


Figure 11: Pareto Efficient Allocations

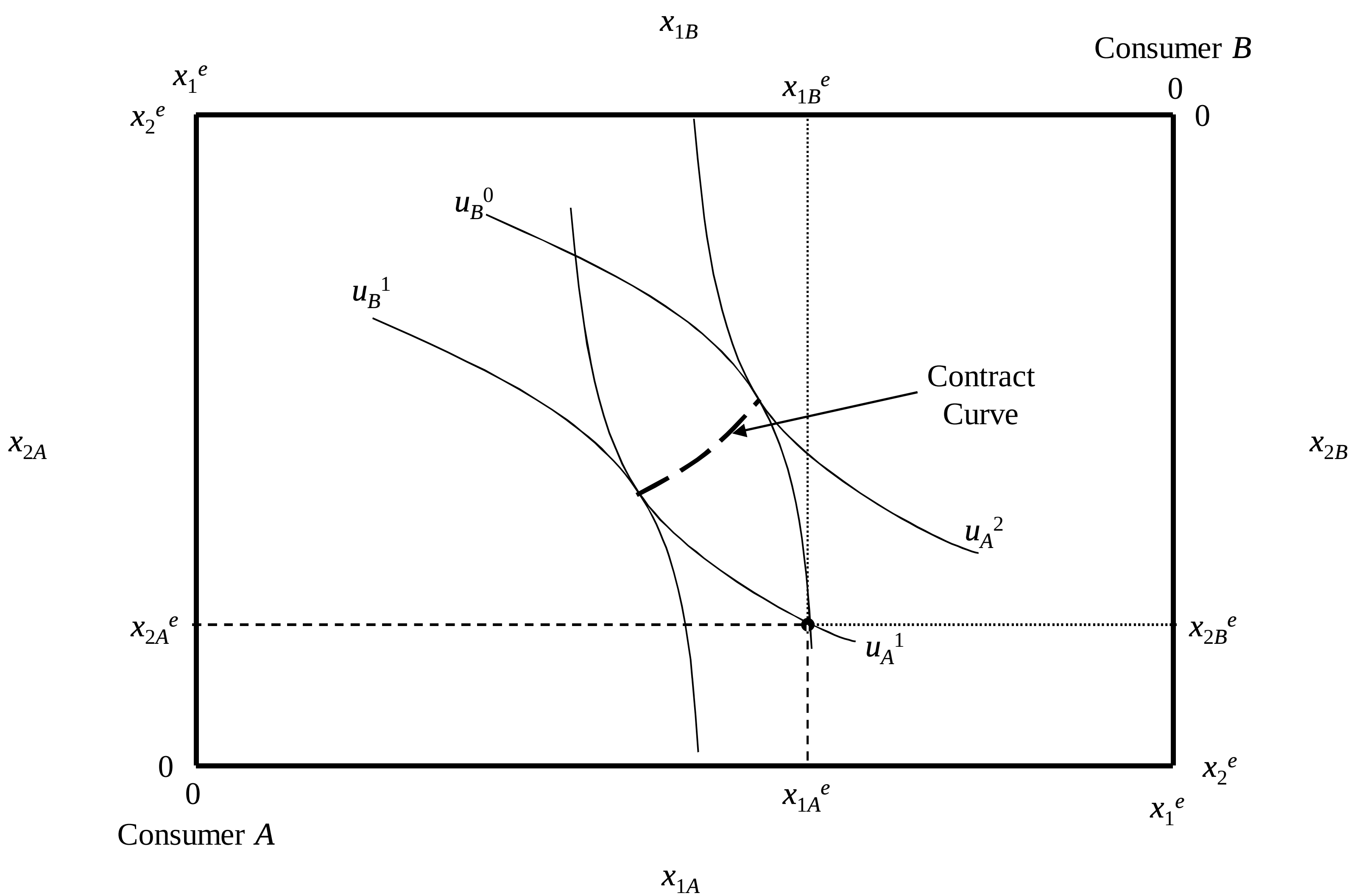


Figure 12: First Welfare Theorem

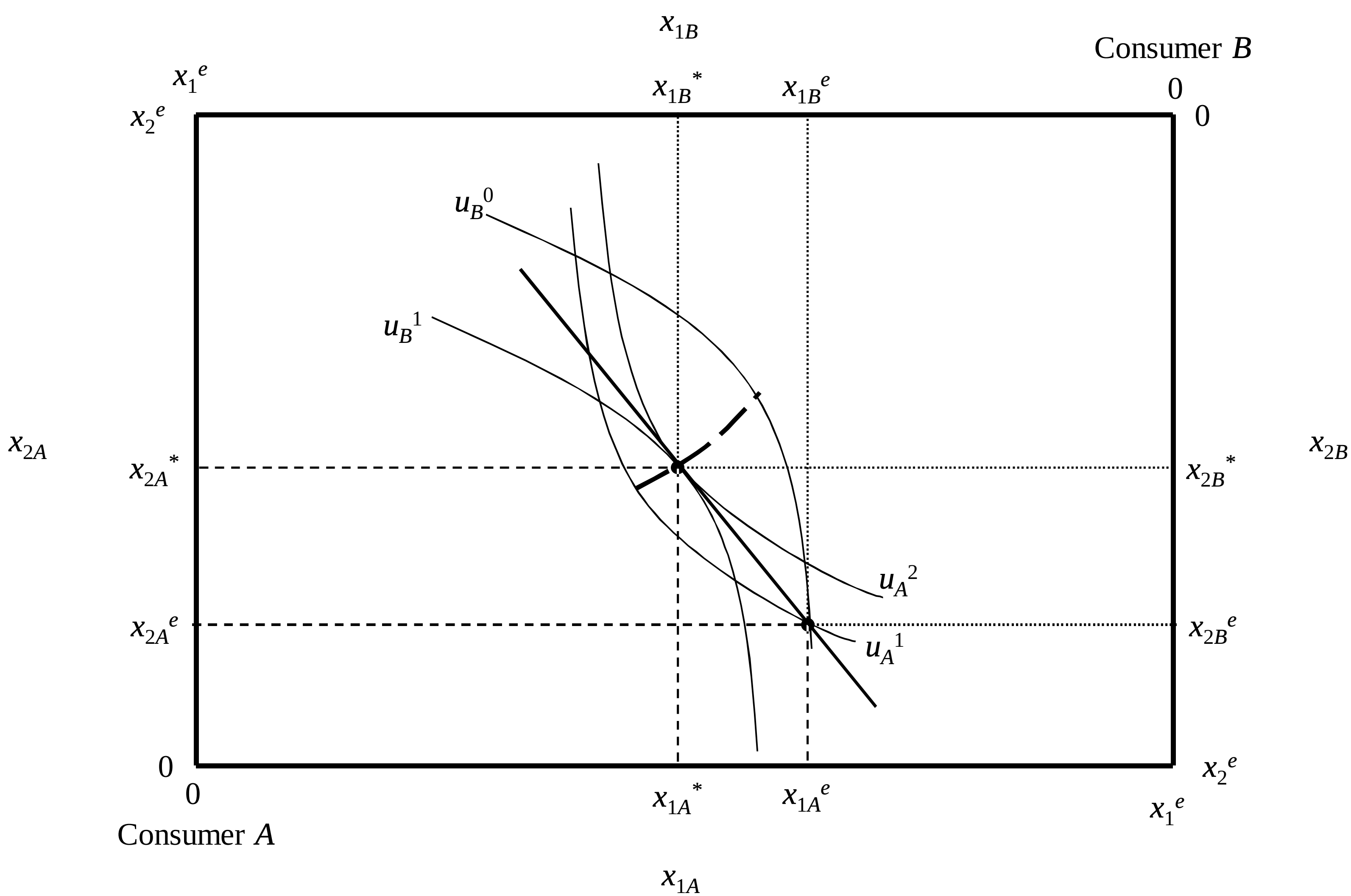


Figure 13: Second Welfare Theorem