DUALITY & INTEGRABILITY

ECON 8001-2 **Instructor: Terry Hurley**

DUALTIY

It is clear that the utility maximization and expenditure minimization problems do not yield the same demand relationship. It should also be clear that there are striking similarities in how the two problems are setup and solved. Therefore, while the solutions are not the same, there is a systematic relationship between the two that economists have found very useful. This relationship is often referred to as duality.

Questions: Why aren't the Marshallian Demand and Hicksian Demand equal in general?
Under what circumstances, if any, will they be equal?

While utility and wealth may be inextricably linked, they are distinct. There are often many different levels of wealth capable of achieving a particular level of utility. Likewise, there are often many different levels of utility that can be achieved for any particular level of wealth.

There are however circumstances when the two will be equal and these circumstance occur systematically where utility is maximized subject to a budget constraint and expenditures are minimized subject to a utility constraint.

PROPOSITION D1: Suppose $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and convex preference relation $\succsim$ on $X = \Re^L_+$; $p >> 0$ is a vector of prices; $u^* = v(p, w^*) > u(0)$; and $w^* = e(p, u^*) > 0$, then $x(p, w^*) = h(p, u^*)$.

To gain some insight into why this result is true, let us compare first order conditions for the two problems assuming an interior solution exists. For UMP, we have

D1 $\dfrac{\partial u(x^*)}{\partial x_l} = l^* p_l$ and $x_l^* > 0$ for $l = 1,\ldots L$, and

D2 $w = p\cdot x^*$, and $l^* > 0$.

For EMP, we have

D3 $p_l = g^* \dfrac{\partial u(h^*)}{\partial h_l}$, and $h_l^* > 0$ for $l = 1,\ldots L$, and

D4 $u = u(h^*)$, and $g^* > 0$.

For any two commodities l and k, equations D1 and D3 imply

D5 $\dfrac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \dfrac{p_l}{p_k}$ and

D6 $$\frac{\frac{\partial u(h^*)}{\partial h_l}}{\frac{\partial u(h^*)}{\partial h_k}} = \frac{p_l}{p_k}.$$

Now if $u^* = v(p, w^*)$ and $w^* = e(p, u^*)$, then equations D2 and D4 imply

D7 $p \cdot h^* = e(p, u^*) = w^* = p \cdot x^*$ and

D8 $u(x^*) = v(p, w^*) = u^* = u(h^*)$.

Notice that all of these equations must be satisfied if $x^* = h^*$. But equations D1 and D2 are the first order conditions for the UMP where $w = e(p, u^*)$ and equations D3 and D4 are the first order conditions for the EMP where $u = v(p, w^*)$.

The importance of PROPOSITION D1 is that it says the Marshallian and Hicksian Demands do yield the same solution to the consumer's problem provided they are evaluated at the right level of wealth and utility (and the same prices). This level of utility and wealth must be mutually consistent. That is, the utility level we evaluate the Hicksian Demand at should equal the maximum amount of utility an individual can achieve from the *UMP* problem given prices and wealth; and the wealth level we evaluate the Marshallian Demand at should equal the minimum expenditure that can be achieved from the *EMP* problem given prices and utility. We will see that the proposition is also useful because it allows us to easily derive other important results about the characteristics of demand.

IMPLICATIONS OF DUALITY

Previously, we showed that if we know the expenditure function it is easy enough to derive Hicksian Demand by simply differentiating with respect to the price. We did not show a similar relationship for Marshallian Demand even though one exists. This relationship is straightforward to see when we can take advantage of PROPOSITION D1. PROPOSITION D1 implies $u^* = v(p, e(p, u^*))$. If we totally differentiate with respect to p_l, we get

D9 $$\frac{\partial v(p, e(p, u^*))}{\partial p_l} + \frac{\partial v(p, e(p, u^*))}{\partial w}\frac{\partial e(p, u^*)}{\partial p_l} = 0.$$

Recall now that $h_l(p, u^*) = \frac{\partial e(p, u^*)}{\partial p_l}$ by PROPOSITION EM4, but PROPOSITION D1 also implies $h_l(p, u^*) = x_l(p, w^*)$. By substituting into equation D9 and rearranging, we get

D10 $$x_l(p,w^*) = -\frac{\frac{\partial v(p,w^*)}{\partial p_l}}{\frac{\partial v(p,w^*)}{\partial w}}.$$

This result is known as Roy's Identity:

PROPOSITION D2: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ on $X = \Re^L_+$. Suppose also that the indirect utility function is differentiable at $(p, w) >> 0$. Then

$$x_l(p, w) = -\frac{\frac{\partial v(p,w)}{\partial p_l}}{\frac{\partial v(p,w)}{\partial w}} \text{ for all } l = 1,\ldots,L \text{ or}$$

$$x(p, w) = -\frac{1}{\nabla_w v(p,w)} \nabla_p v(p,w).$$

Recall that own-price effects for Marshallian Demand were indeterminate, while own-price effects for Hicksian Demand must be negative. PROPOSITION D1 helps us explain what is going on here by facilitating the derivation of what is know as the Slutsky Equation. Note that PROPOSITION D1 implies that $h_l(p, u^*) = x_l(p, e(p, u^*))$. If we differentiate with respect to p_k, we get

D11 $$\frac{\partial h_l(p,u^*)}{\partial p_k} = \frac{\partial x_l(p,e(p,u^*))}{\partial p_k} + \frac{\partial x_l(p,e(p,u^*))}{\partial w}\frac{\partial e(p,u^*)}{\partial p_k}.$$

Note that again PROPOSITION EM4 and PROPOSITION D1 will imply $\frac{\partial e(p,u^*)}{\partial p_k} = h_k(p, u^*)$ $= x_k(p, w^*)$. Substituting this result and the result from PROPOSITION D2, and rearranging equation D11 yields

D12 $$\frac{\partial x_l(p,w^*)}{\partial p_k} = \frac{\partial h_l(p,u^*)}{\partial p_k} - \frac{\partial x_l(p,w^*)}{\partial w} x_k(p,w^*).$$

Intuitively, equation D12 says that there are two effects on Marshallian Demand when prices change. The first effect is a Hicksian substitution effect that will always be negative when $l = k$. The second is a wealth effect that is inversely related to whether the commodity of interest is normal or inferior. It is inversely related because an increase in price decreases the budget set in a way that is analogous to a decrease in wealth. Equation D12 provides an intuitive explanation for why the Marshallian Demand for a commodity can actually increase with its own price and it shows why such circumstances are rare. For such a result to occur, the commodity must be inferior and its positive wealth effect must be greater in absolute magnitude than its negative Hicksian substitution effect.

The implications of equation D12 can be illustrated graphically, which should be familiar to you from your intermediate micro theory class. Figure D1 illustrates the effect of an increase in the price a good 1 from p_1 to p_1'. At the initial price, consumer demand is $x_1(p_1, p_2, w)$ and $x_2(p_1, p_2, w)$ where the indifference curve with utility level $u = v(p_1, p_2, w)$ is just tangent to the budget constraint corresponding to p_1, p_2, and $w = e(p_1, p_2, v(p_1, p_2, w))$. At the new price, consumer demand is $x_1(p_1', p_2, w)$ and $x_2(p_1', p_2, w)$ where the indifference curve with utility level $u' = v(p_1', p_2, w)$ is just tangent to the budget constraint corresponding to p_1', p_2, and $w = e(p_1', p_2, v(p_1', p_2, w))$. To find out how much of the change in consumption is attributable to the income effect and how much is attributable to the Hicksian substitution effect, we can shift the new budget constraint up in a parallel fashion to the budget constraint corresponding to p_1', p_2, and $w = e(p_1', p_2, v(p_1, p_2, w))$ which is just tangent to the initial indifference curve. This tangency corresponds to $h_1(p_1', p_2, v(p_1, p_2, w))$ and $h_2(p_1', p_2, v(p_1, p_2, w))$. The differences $h_1(p_1', p_2, v(p_1, p_2, w)) - x_1(p_1, p_2, w)$ and $h_2(p_1', p_2, v(p_1, p_2, w)) – x_2(p_1, p_2, w)$ which can also be written as $h_1(p_1', p_2, v(p_1, p_2, w)) - h_1(p_1, p_2, v(p_1, p_2, w))$ and $h_1(p_1', p_2, v(p_1, p_2, w)) - h_1(p_1, p_2, v(p_1, p_2, w))$ represent the substitution effect. The differences $x_1(p_1', p_2, w) - h_1(p_1', p_2, v(p_1, p_2, w))$ and $x_2(p_1', p_2, w) – h_2(p_1', p_2, v(p_1, p_2, w))$ which can also be written as $x_1(p_1', p_2, w) – x_1(p_1', p_2, e_2(p_1', p_2, v(p_1, p_2, w)))$ and $x_2(p_1', p_2, w) – x_2(p_1', p_2, e_2(p_1', p_2, v(p_1, p_2, w)))$ represent the income effect.

Note that the Slutsky decomposition illustrated in Figure D1 is for a discrete rather than infinitesimal change in price considered in equation D12. Therefore, we could have also illustrated the decomposition in a slightly different manner (see Figure D2). All intermediate micro theory texts that I have seen use the decomposition in Figure D1 rather than Figure D2, but that doesn't mean the decomposition illustrated in Figure D2 is incorrect. So what is the difference? The difference relates to whether you choose to evaluate the income effect at the new prices as in Figure D1 or at the old prices as in Figure D2. Later, when we explore the welfare effects of price changes, we will run into a similar issue.

The Slutsky result tells us even more about the relationship between Marshallian and Hicksian Demand. If we consider own price effects and know that Marshallian Demand is normal for all p and w, then $\frac{\partial h_k(p,u^*)}{\partial p_k} > \frac{\partial x_k(p,w^*)}{\partial p_k}$, which means the Hicksian Demand is always steeper than the Marshallian Demand or the Marshallian Demand is more price responsive than the Hicksian Demand. Alternatively, if we know Marshallian Demand is inferior for all p and w, then $\frac{\partial x_k(p,w^*)}{\partial p_k} > \frac{\partial h_k(p,u^*)}{\partial p_k}$, which means the Marshallian Demand is always steeper than the Hicksian Demand or the Hicksian Demand is more price responsive than the Marshallian Demand.

A complete statement of the Slutsky Equation is summarized in the next proposition:

PROPOSITION D3: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ on $X = \Re^L_+$. Then for all p, w, and $u = v(p, w)$,

$$\frac{\partial x_l(p,w)}{\partial p_k} = \frac{\partial h_l(p,u)}{\partial p_k} - \frac{\partial x_l(p,w)}{\partial w} x_k(p,w) \qquad \text{for all } l, k = 1,\ldots,L \text{ or}$$

$$D_p x(p, w) = D_p h(p, u) - D_w x(p, w)\, x(p, w)^T.$$

An important corollary that follows immediately from this result and PROPOSITION EM5 is

COROLLARY D3: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ on $X = \Re^L_+$. Then for all p, and w

(i) $S(p, w) = D_p x(p, w) + D_w x(p, w)\, x(p, w)^T$ is negative semidefinite,
(ii) $S(p, w)$ is symmetric, and
(iii) $S(p, w)\, p = 0$.

Note that $S(p, w)$ is commonly referred to as the Slutsky Substitution Matrix. It was originally conceived through a thought experiment: Suppose we increase the price of a good, but then compensate a consumer with additional wealth so that it is as well off after the price change as it was before. How much would demand change? COROLLARY D3 (i) tells us that taken together the income and substitution effect for an increase in a commodity's own price must be negative. COROLLARY D3 (ii) will prove useful when considering our next question which relates to the possibility of recovering someone's preference relation from their Marshallian Demand. COROLLARY D3 (iii) is just another way of thinking about PROPOSITION EM5 (iv).

There is one more result that duality will help us obtain and that we will find useful later on in terms of understanding welfare analysis.

PROPOSITION D4: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ on $X = \Re^L_+$. If $x_k(p, w)$ is normal (inferior) for all p and w, then $h_k(p, u'') > (<)\, h_k(p, u')$ if $u'' > u'$.

Proof: Suppose $u'' > u'$. Choose w' and w'' such that $w'' > w'$, $v(p, w'') = u''$, and $v(p, w') = u'$. By PROPOSITION D1, $x_k(p, w') = h_k(p, v(p, w'))$ and $x_k(p, w'') = h_k(p, v(p, w''))$. Since $x_k(p, w)$ is normal (inferior) for all p and w, $x_k(p, w'') > (<)\, x_k(p, w')$, such that $h_k(p, v(p, w'')) > (<)\, h_k(p, v(p, w'))$. Note that $h_k(p, v(p, w'')) = h_k(p, u'')$ for all p and w. Similarly, $h_k(p, v(p, w')) = h_k(p, u')$ for all p and w. Therefore, for any $u'' > u'$, $h_k(p, u'') > (<)\, h_k(p, u')$ if $x_k(p, w)$ is normal (inferior). **Q.E.D.**

COROLLARY D4: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$ on $X = \Re^L_+$. If $x_k(p, w)$ is independent of wealth for all p and w, then $h_k(p, u'') = h_k(p, u')$ for all u'' and u'.

INTEGRABLITY

We have spent a lot of time constructing these things called Marshallian and Hicksian Demand based on various assumptions about an individual's underlying preference relation. An interesting question is whether we can go the other way. That is, if we observe Marshallian Demand out in the real world, can we actually use what we observe to reconstruct an individual's preference relation? If we can, then we have a stronger argument for using what we have observed to draw conclusion regarding individual welfare because we know it is possible to link what we observe directly back to an individual's preferences. This question is referred to as the integrability question.

Suppose we have obtained data on an individual's consumption and wealth. For example, many developing and developed countries now conduct detailed household consumption surveys. We use this data to estimate the Marshallian Demand $x(p, w)$. Under what conditions, if any, can we use this demand to construct an individual's underlying preference relation?

If we choose some point on this demand, $x^0 = x(p^0, w^0)$, we can assign it an arbitrary utility level, u^0, because a utility function is not unique. For this demand system, we know there is an expenditure function corresponding to u^0: $e(p, u^0)$. From PROPOSITION D1, we also know that this expenditure function satisfies

I1 $$\frac{\partial e(p,u^0)}{\partial p_l} = x_l\left(p, e(p,u^0)\right) \quad \text{for all } l = 1, \ldots, L, \text{ and}$$

I2 $$w^0 = e(p^0, u^0).$$

Equations I1 and I2 define a partial differential system of equations and initial condition that we can presumably solve under the right circumstances (you will not be asked to do this because I do not consider differential equations as a prerequisite for this class). The most interesting of the necessary and sufficient conditions for a solution to exist is the symmetry and negative semidefiniteness of $S(p, w)$ which COROLLARY D3 tells us will be satisfied in preferences are rational, continuous, locally nonsatiated, and strictly convex.

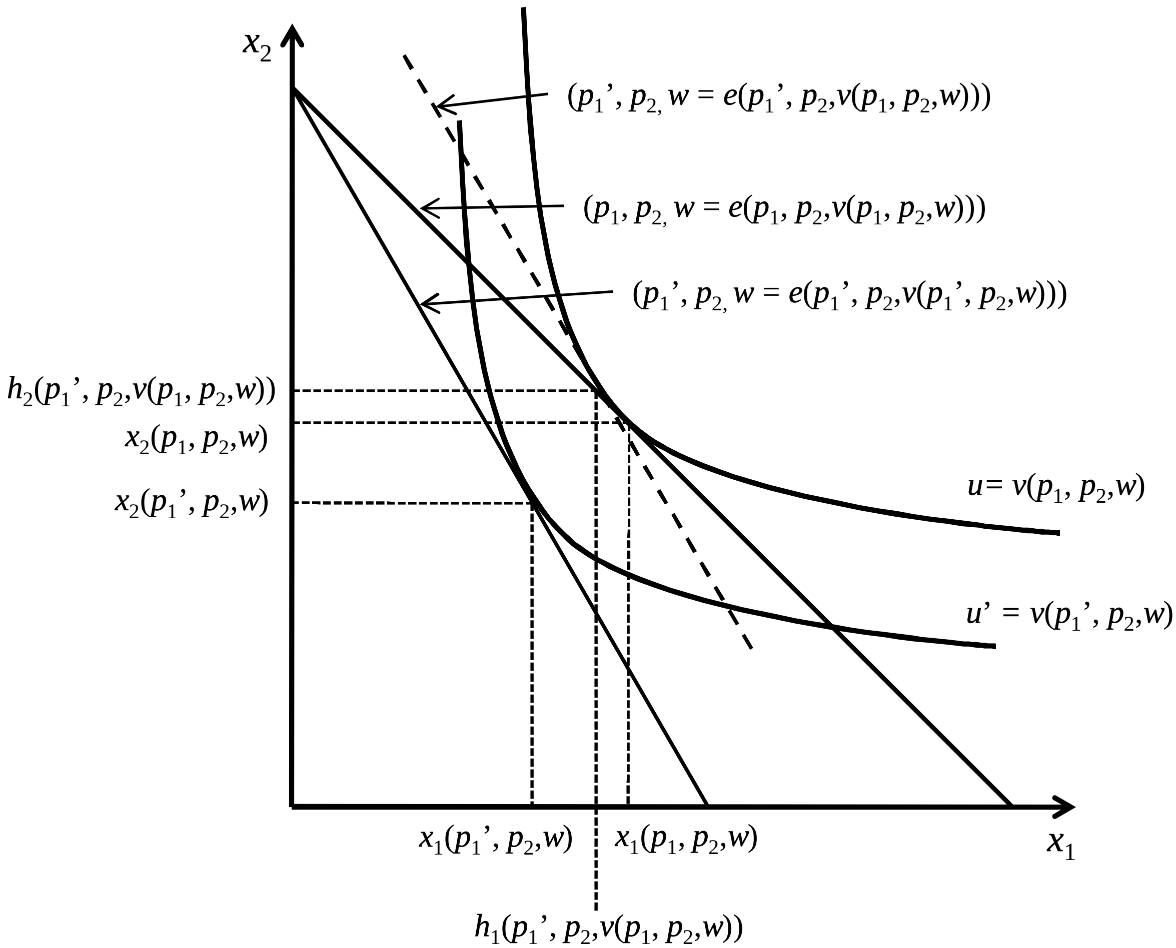
So, now we know how to get an expenditure function, $e(p, u^0)$, corresponding to the utility yielded by consuming $x^0 = x(p^0, w^0)$. But what we really want to do is be able to describe the individual's preference relation. Fortunately, $e(p, u^0)$ provides the information we need to find the set of all vectors x such that $x \succsim x^0$:

I3 $$\{x \in \Re^L_+ : p \cdot x \geq e(p, u^0) \text{ for all } p >> 0\}.$$

Figure I3 illustrates the intuition of Equation I3. Intuitively, as we vary prices, our expenditure function sketches out our indifference curve for u^0 and gradually eliminates all points that fall below u^0. The only points that remain after considering all possible prices are points that are on or above this indifference curve. That is, points that satisfy the condition $x \succsim x^0$.

Let us review what we have just done so everyone understands. We have picked an arbitrary consumption vector. We then used an individual's Marshallian Demand relationship to pick a price vector and level of wealth that is consistent with this arbitrary consumption vector. Next, we constructed the expenditure function that includes this consumption vector and is consistent with our choice of prices and wealth. Finally, we used this expenditure function to identify all other consumption vectors that are preferred or indifferent to our arbitrarily chosen consumption vector. At the end of this whole process, what are we able to say? For all $x \in \Re^L_+$, $x \succsim x^0$ or $x^0 \succ x$. However, if $x', x'' \in \Re^L_+$ and we know $x' \succsim x^0$ and $x'' \succsim x^0$, or $x^0 \succ x''$ and $x^0 \succ x'$, we cannot say anything more about x' and x''. So we only have a partial ordering of preferences. However, we can repeat this exercise as often as necessary in order to construct as much of an individual's preference relation as we need to answer the question of interest. This is a very powerful result that puts us in a position to use estimates of individual demand for welfare analysis.

Figure D1: Graphical illustration of Slutsky income and substitution effects.



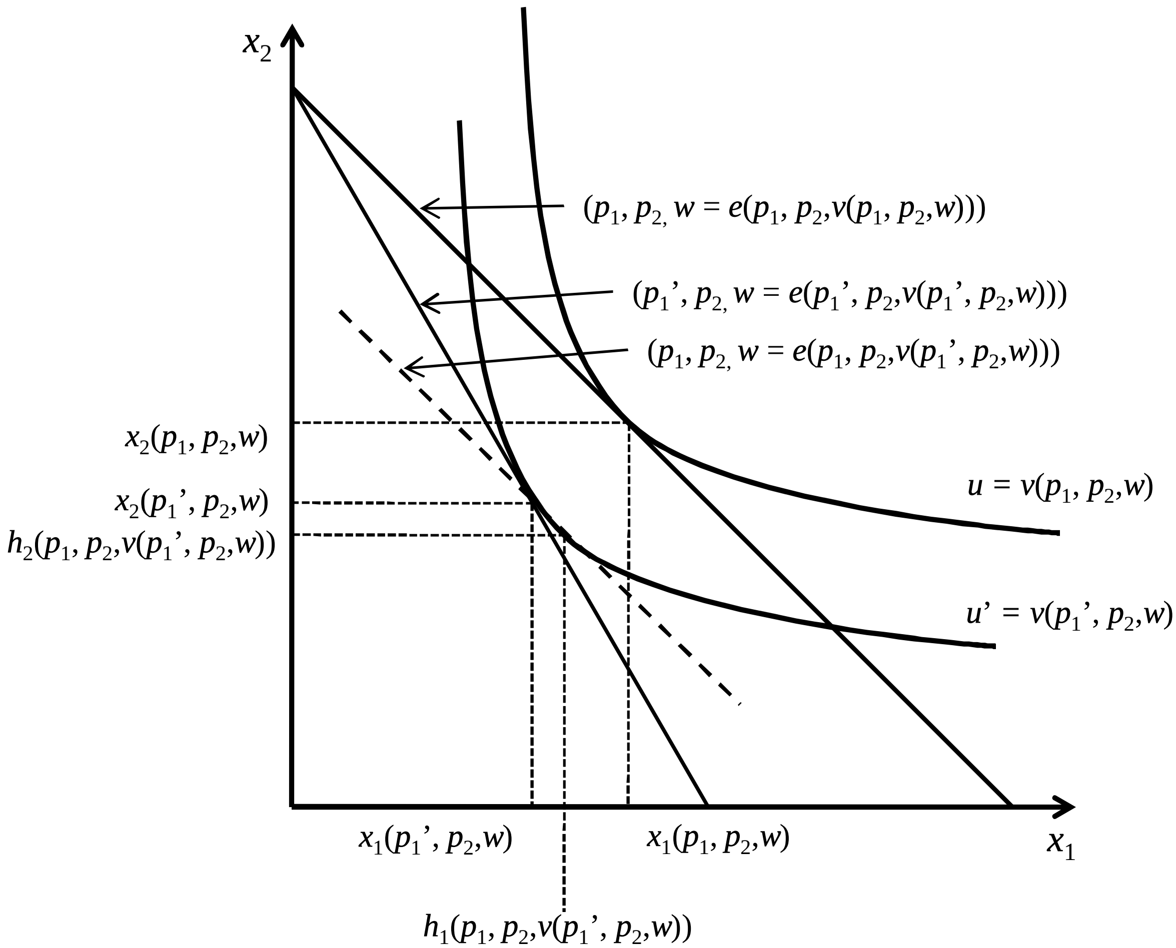
Substitution Effects: $h_1(p_1', p_2, v(p_1, p_2, w)) - x_1(p_1, p_2, w) = h_1(p_1', p_2, v(p_1, p_2, w)) - h_1(p_1, p_2, v(p_1, p_2, w))$

$h_2(p_1', p_2, v(p_1, p_2, w)) - x_2(p_1, p_2, w) = h_2(p_1', p_2, v(p_1, p_2, w)) - h_2(p_1, p_2, v(p_1, p_2, w))$

Income Effects: $x_1(p_1', p_2, w) - h_1(p_1', p_2, v(p_1, p_2, w)) = x_1(p_1', p_2, w) - x_1(p_1', p_2, e(p_1', p_2, v(p_1, p_2, w)))$

$x_2(p_1', p_2, w) - h_2(p_1', p_2, v(p_1, p_2, w)) = x_2(p_1', p_2, w) - x_2(p_1', p_2, e(p_1', p_2, v(p_1, p_2, w)))$

Figure D2: Alternative illustration of Slutsky income and substitution effects.



Substitution Effects: $x_1(p_1', p_2,w) - h_1(p_1, p_2,v(p_1', p_2,w)) = h_1(p_1', p_2,v(p_1', p_2,w)) - h_1(p_1, p_2,v(p_1', p_2,w))$
$x_2(p_1', p_2,w) - h_2(p_1, p_2,v(p_1', p_2,w)) = h_2(p_1', p_2,v(p_1', p_2,w)) - h_2(p_1, p_2,v(p_1', p_2,w))$

Income Effects: $h_1(p_1, p_2,v(p_1', p_2,w)) - x_1(p_1, p_2,w) = x_1(p_1, p_2, e(p_1, p_2,v(p_1', p_2,w))) - x_1(p_1, p_2,w)$
$h_2(p_1, p_2,v(p_1', p_2,w)) - x_2(p_1, p_2,w) = x_2(p_1, p_2, e(p_1, p_2,v(p_1', p_2,w))) - x_2(p_1, p_2,w)$

Figure I1: Using the expenditure function to map the "at least as good as" set for $x \in \Re_+^2$