COST MINIMIZATION

ECON 8001-2 **Instructor: Terry Hurley**

INTRODUCTION

We begin our analysis of how to produce and how much to produce by looking at the classical cost minimization problem, which will tell us how to produce assuming we know how much we want to produce. There are two key assumptions for the classical cost minimization problem aside from those we will make about the nature of the production possibilities set: cost minimization and competitive factor markets. The cost minimization assumption essentially says that producers will determine how to produce by minimizing their factor costs. The competitive factor market assumption says that a producer's choice of factors does not affect the price of those factors. There are certainly examples where both these assumptions may make little sense. Still, there are plenty of examples where they are quite sensible and they do provide a useful benchmark even when they do not make sense.

SINGLE OUTPUT & MANY FACTORS

With a single output and many factors, the classic cost minimization problem for a competitive producer can be written as

CM1 $$\min_{z_1 \geq 0, \ldots, z_N \geq 0} r \cdot z \text{ subject to } z \in Z(q).$$

Note that the producer's choice of factors is constrained by the Factor Requirement Set. If it is possible to describe $Z(q)$ using a nice differentiable production function $f(z)$, then we can rewrite this constraint as $f(z) \geq q$, set up a Lagrangian, and derive the first-order necessary conditions:

CM2 $$L = r \cdot z + g\left(q - f(z)\right),$$

CM3 $$\frac{\partial L}{\partial z_n} = r_n - g^* \frac{\partial f\left(z^*\right)}{\partial z_n} \geq 0,\ \frac{\partial L}{\partial z_n} z_n^* = 0, \text{ and } z_n^* \geq 0 \text{ for } n = 1,\ldots, N,$$

CM4 $$\frac{\partial L}{\partial g} = q - f\left(z^*\right) \leq 0,\ \frac{\partial L}{\partial g} g^* = 0, \text{ and } g^* \geq 0.$$

With the assumption of a monotonic Factor Requirement Set, it is straightforward to show that production for cost minimization will be efficient: $f(z^*) = q^*$ (this is analogous to no excess utility property for the consumer's expenditure minimization problem). With a Strictly Convex Factor Requirement Set, it can be shown that a unique optimum will exist, which is analogous to the strict convexity assumption giving us a unique optimum for the consumer's expenditure minimization problem. For positive output, it must be the case that $g^* = \frac{r_n}{\frac{\partial f\left(z^*\right)}{\partial z_n}}$ for some n

where $\dfrac{r_n}{\frac{\partial f(z^*)}{\partial z_n}}$ reflects the *Marginal Cost of Production* (we will show this formally below). If multiple factors are used, then it must also be true that $\dfrac{r_n}{\frac{\partial f(z^*)}{\partial z_n}} = \dfrac{r_m}{\frac{\partial f(z^*)}{\partial z_m}}$ or $\dfrac{r_n}{r_m} = \dfrac{\frac{\partial f(z^*)}{\partial z_n}}{\frac{\partial f(z^*)}{\partial z_m}}$ for any *n* and *m* used in production. The first of these expressions says that the Marginal Cost of Production is equal across factors, while the second says the MRTS must equal the ratio of factor prices. The consumer problem analogue is the equality of the Marginal Rate of Substitution and ratio of commodity prices.

The solution to this problem will be a set of *Conditional Factor Demands* that depend on the price of factors and output: $z(r, q)$. Like with the Hicksian demands for the consumer problem, it is possible to show that the Conditional Factor Demands have some useful properties.

The first is that they are homogeneous of degree zero in factor prices. This should be easy to see by looking at how multiplying *r* by some $a > 0$ changes the conditions for an optimum implied by equation CM3.

Second, if we have an interior solution and totally differentiate equation CM3 and CM4 with respect to g^*, z^*, q, and r we get

CM5
$$\begin{bmatrix} 0 & -\frac{\partial f(z^*)}{\partial z_1} & \cdots & -\frac{\partial f(z^*)}{\partial z_N} \\ -\frac{\partial f(z^*)}{\partial z_1} & -g^*\frac{\partial^2 f(z^*)}{\partial z_1^{\,2}} & \cdots & -g^*\frac{\partial^2 f(z^*)}{\partial z_1 \partial z_N} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial f(z^*)}{\partial z_N} & -g^*\frac{\partial^2 f(z^*)}{\partial z_N \partial z_1} & \cdots & -g^*\frac{\partial^2 f(z^*)}{\partial z_N^{\,2}} \end{bmatrix} \begin{bmatrix} dg^* \\ dz_1^* \\ \vdots \\ dz_N^* \end{bmatrix} = H \begin{bmatrix} dg^* \\ dz_1^* \\ \vdots \\ dz_N^* \end{bmatrix} = -I \begin{bmatrix} dq \\ dr_1 \\ \vdots \\ dr_N \end{bmatrix},$$

where *H* is referred to as the bordered Hessian matrix and *I* is a $N + 1 \times N + 1$ identity matrix. Solving equation CM5 we can get the Conditional Factor Demand Substitution Matrix by discarding the first row and column,

CM6
$$\begin{bmatrix} \frac{\partial z_1(r,q)}{\partial r_1} & \cdots & \frac{\partial z_1(r,q)}{dr_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_N(r,q)}{\partial r_1} & \cdots & \frac{\partial z_N(r,q)}{\partial r_N} \end{bmatrix},$$

which is symmetric and negative semidefinite, so own price effects are nonpositive.

To explore the generality of this result, let $z^0 = z(r^0, q)$ and $z^1 = z(r^1, q)$. By the definition of an optimum, it must be true that

CM7 $r^0 \cdot z(r^0, q) \leq r^1 \cdot z(r^0, q)$ and

CM8 $r^1 \cdot z(r^1, q) \leq r^0 \cdot z(r^1, q)$

implying

CM9 $(r^0 - r^1) \cdot (z(r^0, q) - z(r^1, q)) \leq 0$ or

CM9' $\Delta r \cdot \Delta z \leq 0$.

Equation CM9' generalizes the result that own price effects for Conditional Factor Demands must be nonpositive. For example, if the only change in price is an increase in the price of factor n, then $\Delta r_n \Delta z_n \leq 0$, which means $0 \geq \Delta z_n$, so Conditional Factor Demands are nonincreasing in their own price. This result is more general because it will hold anytime we have a minimum. Therefore, the condition also provides an opportunity to test the validity of the cost minimization assuming factor prices change but output does not. However, there is a more general way to test the cost minimization assumption if changes in factor prices and output are observed.

DEFINITION

Weak Axiom of Cost Minimization (*WACM*): If z' is a cost minimizing vector of factors given factor prices r' and output q', then $r' \cdot z' \leq r' \cdot z$ for all $q \geq q'$ and $z \in Z(q)$.

WACM provides a set of conditions that observed factor prices and outputs from a producer must satisfy if the producer is indeed minimizing costs.

Substituting the Conditional Factor Demands into the objective function yields what is referred to as the *Cost Function*:

CM10 $c(r, q) = r \cdot z(r, q)$.

This Cost Function is

(i) nondecreasing in factor prices (r) and output (q);
(ii) homogeneous of degree 1 in r; and
(iii) concave in r.

These properties and their proof are similar to the expenditure function, so we will not go into them in more detail here. These properties will also hold more generally, so we will formalize them after talking about the multiple output and multiple factor case.

Another property that is identical to the expenditure function is

(iv) $z_n(r, q) = \dfrac{\partial c(r,q)}{\partial r_n}$ for $n = 1,\ldots,N$,

which is often referred to as Shephard's Lemma.

Also recall that under the right conditions, integrability said we could use our Marshallian demand to get an expenditure function and use this expenditure function to recover an individual's preference relation. It turns out that if we know our input requirement set is convex, then we can use our cost function to recover our production possibilities set: $Z(q) = \{z \in \Re^N_+ : r \cdot z \geq c(r, q) \text{ for all } r >> 0\}$.

Finally, note that

CM11 $$c(r, q) \equiv r \cdot z(r,q) + g(r,q)(q - f(z(r,q))).$$

Differentiating with respect to q yields

CM12 $$\frac{\partial c(r,q)}{\partial q}dq = \sum_{n=1}^{N} r_n \frac{\partial z_n(r,q)}{\partial q}dq + \frac{\partial g(r,q)}{\partial q}(q - f(z(r,q)))dq + g(r,q)\left(1 - \sum_{n=1}^{N}\frac{\partial f(z(r,q))}{\partial z_n}\frac{\partial z_n(r,q)}{\partial q}\right)dq.$$

For an interior solution, equations CM3 and CM4 imply $\dfrac{\partial f(z(r,q))}{\partial z_n} = \dfrac{r_n}{g(r,q)}$ and $q = f(z(r,q))$, which substituting into equation CM12 implies

CM13 $$\frac{\partial c(r,q)}{\partial q} = \sum_{n=1}^{N} r_n \frac{\partial z_n(r,q)}{\partial q} + g(r,q) - \sum_{n=1}^{N} r_n \frac{\partial z_n(r,q)}{\partial q} = g(r,q).$$

The left-hand side of equation CM13 is the Marginal Cost of Production, which we can refer to as $mc(r, q)$. This marginal cost of production is equal to the Lagrange multiplier evaluate at the optimum, which we know from above equals $\dfrac{r_n}{\frac{\partial f(z(r,q))}{\partial z_n}}$ for all n used in production.

Another quantity we will find useful is the average cost of production:

DEFINITION:

Average Cost of Production ($ac(r, q)$): The cost of production divided by output: $ac(r, q) = c(r, q)/q$.

There are some important relationships between average and marginal costs that you should recall. If we can differentiate average costs with respect to q twice, we get

CM14
$$\frac{\partial ac(r,q)}{\partial q} = \frac{\frac{\partial c(r,q)}{\partial q} q - c(r,q)}{q^2} = \frac{\frac{\partial c(r,q)}{\partial q} - ac(r,q)}{q} \text{ and}$$

CM15
$$\frac{\partial^2 ac(r,q)}{\partial q^2} = \frac{\left(\frac{\partial^2 c(r,q)}{\partial q^2} - \frac{\partial ac(r,q)}{\partial q}\right) q - \left(\frac{\partial c(r,q)}{\partial q} - ac(r,q)\right)}{q^2}.$$

If average cost is at a minimum q^{min}, equation CM14 must equal 0 and equation CM15 must be non-negative, which implies

CM14'
$$\frac{\partial c(r,q^{\min})}{\partial q} = ac(r,q^{\min}) \text{ and}$$

CM15'
$$\frac{\partial^2 c(r,q^{\min})}{\partial q^2} \geq 0.$$

Equation CM14' says that marginal cost equals average cost at the minimum average cost if the marginal cost is nondecreasing.

LONG-RUN VS SHORT-RUN WITH SINGLE OUTPUT & MANY FACTORS

Let $z = (z_v, z_f)$ where z_v is a vector of factors that we will refer to as variable factors and z_f is a vector of factors that we will refer to as fixed factors. Let $r = (r_v, r_f)$ where r_v is a vector of prices for variable factors and r_f is a vector of prices for fixed factors. Finally, let $q = f(z_v, z_f)$ be the production function. Assume a producer can only vary z_v in the Short-Run, but can vary z_v and z_f in the Long-Run. We can write the producer's Short-Run cost minimization problem as

CM16
$$\min_{z_v \geq 0} r_v \cdot z_v + r_f \cdot z_f \text{ subject to } q \leq f(z_v, z_f).$$

The solution to this problem will be a set of *Short-Run Conditional Factor Demands* that will depend on variable factor prices, output, and the vector of fixed factors: $z_v(r_v, q, z_f)$. We can use these Short-Run Factor Demands to define a variety of different cost functions.

DEFINITIONS

Short-Run Total Cost: $STC(r_v, r_f, q, z_f) = r_v \times z_v(r_v, q, z_f) + r_f \times z_f$

Short-Run Variable Cost: $SVC(r_v, q, z_f) = r_v \times z_v(r_v, q, z_f)$

Short-Run Fixed Cost: $SFC(r_f, z_f) = r_f \times z_f$

Short-Run Average Total Cost: $SATC(r_v, r_f, q, z_f) = STC(r_v, r_f, q, z_f)/q$

Short-Run Average Variable Cost: $SAVC(r_v, q, z_f) = SVC(r_v, q, z_f)/q$

Short-Run Average Fixed Cost: $SAFC(r_f, z_f) = SFC(r_f, z_f)/q$

Short-Run Marginal Cost: $SMC(r_v, q, z_f) = \dfrac{\partial STC(r_v, r_f, q, z_f)}{\partial q} = \dfrac{\partial SVC(r_v, q, z_f)}{\partial q}$

The Long-Run cost minimization problem is

CM17 $$\min_{z_v \geq 0, z_f \geq 0} r_v \cdot z_v + r_f \cdot z_f \text{ subject to } q \leq f(z_v, z_f).$$

Note that this is actually the same problem as we did originally. The solution to this problem will be a set of *Long-Run Conditional Factor Demands* that will depend on factor prices and output: $z_v(r_v, r_f, q)$ and $z_f(r_v, r_f, q)$. We can use these Long-Run Factor Demands to define some additional cost functions.

DEFINITIONS

Long-Run Total Cost: $LTC(r_v, r_f, q) = r_v \times z_v(r_v, r_f, q) + r_f \times z_f(r_v, r_f, q)$

Long-Run Average Total Cost: $LATC(r_v, r_f, q) = LTC(r_v, r_f, q)/q$

Long-Run Marginal Cost: $LMC(r_v, r_f, q) = \dfrac{\partial LTC(r_v, r_f, q)}{\partial q}$

Comparing Short and Long-Run Costs, it must be true that $STC(r_v, r_f, q, z_f) \geq LTC(r_v, r_f, q)$ and $SATC(r_v, r_f, q, z_f) \geq LATC(r_v, r_f, q)$ because we can always choose the same factors in the long-run as we used in the short-run.

Define

CM18 $$g(r_v, r_f, q, z_f) = STC(r_v, r_f, q, z_f) - LTC(r_v, r_f, q) \geq 0,$$

and set $z_f = z_f^0 = z_f(r_v, r_f, q)$ such that $g(r_v, r_f, q, z_f^0) = 0$ is at its minimum. Differentiating with respect to q yields

CM19 $$\frac{\partial STC\left(r_v, r_f, q, z_f^{\ 0}\right)}{\partial q} - \frac{\partial LTC\left(r_v, r_f, q\right)}{\partial q} = 0 \text{ and}$$

CM20 $$\frac{\partial^2 STC\left(r_v, r_f, q, z_f^{\ 0}\right)}{\partial q^2} - \frac{\partial^2 LTC\left(r_v, r_f, q\right)}{\partial q^2} \geq 0.$$

Equations CM19 and CM20, imply

CM19' $$SMC\left(r_v, r_f, q, z_f^{\ 0}\right) = LMC\left(r_v, r_f, q\right) \text{ and}$$

CM20' $$\frac{\partial SMC\left(r_v, r_f, q, z_f^{\ 0}\right)}{\partial q} \geq \frac{\partial LMC\left(r_v, r_f, q\right)}{\partial q},$$

which means the Short-Run marginal cost curve must intersect the Long-Run marginal cost curve from below.

Differentiating $g(r_v, r_f, q, z_f^0)$ with respect to the nth variable input price yields

CM21 $$\frac{\partial STC\left(r_v, r_f, q, z_f^{\ 0}\right)}{\partial r_{v_n}} - \frac{\partial LTC\left(r_v, r_f, q\right)}{\partial r_{v_n}} = 0 \text{ and}$$

CM22 $$\frac{\partial^2 STC\left(r_v, r_f, q, z_f^{\ 0}\right)}{\partial r_{v_n}^{\ 2}} - \frac{\partial^2 LTC\left(r_v, r_f, q\right)}{\partial r_{v_n}^{\ 2}} \geq 0.$$

Equations CM21 and CM22, and the fact that factor demands are nonincreasing in own prices, then imply

CM21' $$z_{v_n}\left(r_v, r_f, q, z_f^{\ 0}\right) = z_{v_n}\left(r_v, r_f, q\right) \text{ and}$$

CM22' $$0 \geq \frac{\partial z_{v_n}\left(r_v, r_f, q, z_f^{\ 0}\right)}{\partial r_{v_n}} \geq \frac{\partial z_{v_n}\left(r_v, r_f, q\right)}{\partial r_{v_n}}.$$

which means the Long-Run Conditional Factor Demand is more elastic than the Short-Run Conditional Factor Demand when evaluated at the long-run optimum (note that the Conditional Factor Demand Elasticity is defined generally as $e_{z_n} = \frac{\partial z_n}{\partial r_n}\frac{r_n}{z_n}$).

EXAMPLES WITH SINGLE OUTPUT & MANY FACTORS

We will finish up this single output many factors case with a couple of examples. The first example uses the Leontief Factor Requirement Set: $Z(q) = \{(z_1, z_2) \in \Re_+^2 : \min\{a_1 z_1, a_2 z_2\}^b \geq q\}$ where $a_1 > 0$, $a_2 > 0$, and $b > 0$. The price of factors z_1 and z_2 are $r_1 > 0$ and $r_2 > 0$. With this technology, the cost minimization problem is

CM23 $$\min_{z_1 \geq 0, z_2 \geq 0} r_1 z_1 + r_2 z_2 \text{ subject to } z \in Z(q).$$

Unfortunately, standard differential calculus techniques are not applicable here because the Factor Requirement Set does not give us a nicely differentiable production function. So, where do we go from here? Notice that if $z_1^1 > z_1^0$ and $z_2^1 \geq z_2^0$, or $z_1^1 \geq z_1^0$ and $z_2^1 > z_2^0$, then $\min\{a_1 z_1^1, a_2 z_2^1\}^b \geq \min\{a_1 z_1^0, a_2 z_2^0\}^b$, which means the Input Requirement Set is monotonic. Monotonicity implies that production will be efficient. Since we are assuming the price of all factors is strictly positive, efficiency then implies $a_1 z_1^* = a_2 z_2^*$ and $\min\{a_1 z_1^*, a_2 z_2^*\}^b = q$. Therefore,

CM24 $$z_1^* = \frac{q^{\frac{1}{b}}}{a_1} \text{ and}$$

CM25 $$z_2^* = \frac{q^{\frac{1}{b}}}{a_2},$$

which are our Conditional Factor Demands. The cost function is then

CM26 $$c(r_1, r_2, q) = r_1 \frac{q^{\frac{1}{b}}}{a_1} + r_2 \frac{q^{\frac{1}{b}}}{a_2} = \left(\frac{r_1 a_2 + r_2 a_1}{a_1 a_2}\right) q^{\frac{1}{b}}.$$

Note that for any $t > 0$ $c(tr_1, tr_2, q) = \left(\frac{tr_1 a_2 + tr_2 a_1}{a_1 a_2}\right) q^{\frac{1}{b}} = t\left(\frac{r_1 a_2 + r_2 a_1}{a_1 a_2}\right) q^{\frac{1}{b}} = tc(r_1, r_2, q)$, so our cost function is homogeneous of degree one in factor prices. Note that $\frac{\partial c(r_1, r_2, q)}{\partial r_i} = \frac{q^{\frac{1}{b}}}{a_i} = z_i(r_1, r_2, q) \geq 0$ (i = 1, 2) and $\frac{\partial c(r_1, r_2, q)}{\partial q} = \frac{1}{b}\left(\frac{r_1 a_2 + r_2 a_1}{a_1 a_2}\right) q^{\frac{1}{b}-1} \geq 0$ for $r >> 0$ and $q \geq 0$, so our cost function is nondecreasing in factor price and output, and Shephard's Lemma holds. Finally

note that $H = \begin{bmatrix} \frac{\partial^2 c(r_1,r_2,q)}{\partial r_1^{\,2}} & \frac{\partial^2 c(r_1,r_2,q)}{\partial r_1 \partial r_2} \\ \frac{\partial^2 c(r_1,r_2,q)}{\partial r_2 \partial r_1} & \frac{\partial^2 c(r_1,r_2,q)}{\partial r_2^{\,2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, such that $\begin{vmatrix} 0-e & 0 \\ 0 & 0-e \end{vmatrix}$ implies $e^2 = 0$.
Therefore, our eigenvalues are nonpositive and H is negative semidefinite such that our cost function is concave in factor prices.

For our second example, consider the cost function $c(r_1, r_2, q) = r_1^{\,a} r_2^{\,1-a} q^b$ where $1 > a > 0$ and $b > 0$. We will skip the exercise of showing that this cost function satisfies the properties described above (it does). Instead, we will ask ourselves what production function is consistent with this cost function. In a two factor case like this one, answering this question is actually quite straightforward. Shephard's Lemma implies

CM27 $$z_1(r_1, r_2, q) = \frac{\partial c(r_1, r_2, q)}{\partial r_1} = a\left(\frac{r_2}{r_1}\right)^{1-a} q^b \text{ and}$$

CM28 $$z_2(r_1, r_2, q) = \frac{\partial c(r_1, r_2, q)}{\partial r_2} = (1-a)\left(\frac{r_1}{r_2}\right)^{a} q^b .$$

Note that equation CM27 implies $\frac{r_1}{r_2} = \left(\frac{aq^b}{z_1}\right)^{\frac{1}{1-a}}$. Substituting this result back into equation CM28 then yields $z_2 = (1-a)\left(\left(\frac{aq^b}{z_1}\right)^{\frac{1}{1-a}}\right)^{a} q^b$ or $q = f(z_1, z_2) = \left(\frac{z_1^{\,a} z_2^{\,1-a}}{(1-a)^{1-a} a^a}\right)^{\frac{1}{b}}$, which is the production relationship we are looking for. To verify this note that $\frac{\frac{\partial f(z_1,z_2)}{\partial z_1}}{\frac{\partial f(z_1,z_2)}{\partial z_2}} =$

$$\frac{\frac{1}{b}\left(\frac{z_1^{\,a} z_2^{\,1-a}}{(1-a)^{1-a} a^a}\right)^{\frac{1}{b}-1} \frac{a z_1^{\,a-1} z_2^{\,1-a}}{(1-a)^{1-a} a^a}}{\frac{1}{b}\left(\frac{z_1^{\,a} z_2^{\,1-a}}{(1-a)^{1-a} a^a}\right)^{\frac{1}{b}-1} \frac{(1-a) z_1^{\,a} z_2^{\,1-a-1}}{(1-a)^{1-a} a^a}} = \frac{a z_2}{(1-a) z_1}$$ such that $z_2^{\,*} = \frac{(1-a) r_1}{a r_2} z_1^{\,*}$ for an interior optimum. Substituting back into our production relationship then yields

$$q = \left(\frac{z_1^{*a} \left(\frac{(1-a) r_1}{a r_2} z_1^{*} \right)^{1-a}}{(1-a)^{1-a} a^{a}} \right)^{\frac{1}{b}}$$ or $z_1^{*} = a \left(\frac{r_2}{r_1} \right)^{1-a} q^b$ and $z_2^{*} = (1-a)\left(\frac{r_1}{r_2} \right)^{a} q^b$. Our cost function is

then $c(r_1, r_2, q) = r_1 a \left(\frac{r_2}{r_1} \right)^{1-a} q^b + r_2 (1-a) \left(\frac{r_1}{r_2} \right)^{a} q^b = r_1^{a} r_2^{1-a} q^b$, which is right back where we started.

MANY OUTPUTS & MANY FACTORS

Extending our single output and many factors cost minimization problem, to a many outputs and many factors problem is straightforward. To do this let us rewrite our transformation function $T(y)$ as $T(q, -z)$ where q can now be a vector of outputs. The cost minimization problem can then be written as

CM1' $\min_{z_1 \geq 0, \ldots, z_N \geq 0} r \cdot z$ subject to $T(q, -z) = 0$.

Note that here we are taking advantage of the fact that we know production will be efficient assuming either monotonicity or free disposal. If $T(q, -z)$ is differentiable, then we can use the standard Kuhn-Tucker approach for this problem. The Lagrangian and first-order necessary conditions are:

CM2' $L = r \cdot z + g\left(T(q, -z)\right)$,

CM3' $\frac{\partial L}{\partial z_n} = r_n - g^{*} \frac{\partial T(q, -z^{*})}{\partial z_n} \geq 0$, $\frac{\partial L}{\partial z_n} z_n^{*} = 0$, and $z_n^{*} \geq 0$ for $n = 1, \ldots, N$,

CM4' $\frac{\partial L}{\partial g} = T(q, -z) = 0$, and $g^{*} \geq 0$.

If $Z(q)$ is convex, these conditions are necessary and sufficient for the solution z^{*} and l^{*} to be the optimum. The conditional factor demands defined by equations CM2', CM3' and CM4' will depend of the vector of factor prices and outputs: $z^{*} = z(r, q)$. If $z_n^{*} > 0$ and $z_m^{*} > 0$ at this optimum, then $\frac{\frac{\partial T(q, -z^{*})}{\partial z_n}}{\frac{\partial T(q, -z^{*})}{\partial z_m}} = \frac{r_n}{r_m}$. That is the MRTS will equal the ratio of factor prices just as before. Finally, we can also define the cost function just as before: $c(r, q) = r \cdot z(r, q)$.

COST MINIMIZATION

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We will now summarize some useful properties of conditional factors demands and the cost function more formally:

PROPOSITION CM1: If $z(r, q) \in \Re_+^N$ for $r \in \Re_{++}^N$ and $q \in \Re_+^M$ are the conditional factor demands for a technology $(q, -z) \in Y$ that is closed, satisfies free disposal, and has the input requirement set $Z(q)$, then

(i) $z(r, q)$ is a convex/singleton set if $Z(q)$ is convex/strictly convex;

(ii) $z(r, q)$ is homogeneous of degree 0 in r;

(iii) $(r^0 - r^1) \cdot (z(r^0, q) - z(r^1, q)) \leq 0$ for all $r^0, r^1 \in \Re_{++}^N$; and

(iv) if $z(r, q)$ is differentiable in r, $D_r z(r, q) = \begin{bmatrix} \frac{\partial z_1(r,q)}{\partial r_1} & \cdots & \frac{\partial z_1(r,q)}{\partial r_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_N(r,q)}{\partial r_1} & \cdots & \frac{\partial z_N(r,q)}{\partial r_N} \end{bmatrix}$ is symmetric and negative semidefinite, and $D_r z(r, q) \cdot r = 0$.

PROPOSITION CM2: If $c(r, q) \geq 0$ for $r \in \Re_{++}^N$ and $q \in \Re_+^M$ is the cost function for a technology $(q, -z) \in Y$ that is closed, satisfies free disposal, has the conditional factor demands $z(r, q) \in \Re_+^N$, and has the input requirement set $Z(q)$, then

(i) $c(r, q)$ is homogeneous of degree one in r;

(ii) $c(r, q)$ is nondecreasing in r_n for all n and q_m for all m;

(iii) $c(r, q)$ is concave in r;

(iv) if $c(r, q)$ is differentiable in r, then $D_r c(r, q) = z(r, q)$ (Shepard's Lemma);

(v) if $c(r, q)$ is twice differentiable in r, $D_r^2 c(r, q) = \begin{bmatrix} \frac{\partial^2 c(r,q)}{\partial r_1^2} & \cdots & \frac{\partial^2 c(r,q)}{\partial r_N \partial r_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 c(r,q)}{\partial r_1 \partial r_N} & \cdots & \frac{\partial^2 c(r,q)}{\partial r_N^2} \end{bmatrix}$ is symmetric and negative semidefinite and $D_r^2 c(r, q) \cdot r = 0$; and

(vi) if $Z(q)$ is convex, then $Z(q) = \{z \in \Re^N_+ : r \cdot z \geq c(r, q) \text{ for all } r >> 0\}$.

These conditions are analogous to the conditions we had for Hicksian demands and the expenditure function with two exceptions. First, we said our expenditure function was strictly increasing in u, but here the cost function is nondecreasing is q_m. The difference here relates back to subtle differences between the assumptions of local nonsatiation and free disposal. Local nonsatiation says we can always find a nearby bundle that is strictly preferred. Free disposal is not as restrictive, so it is possible to have a technology where an increase in inputs does not necessarily increase an output. Second, we said our expenditure function was continuous, which followed from the assumption that our preference relation was continuous. Note that we have made no continuity assumptions regarding our production possibilities set. Therefore, we cannot guarantee that our cost function is continuous. However, if our production possibilities set is continuous, then our cost function will be continuous.