CLASSICAL CONSUMER PROBLEM
Comparative Static Notes

ECON 8001-2 **Instructor: Terry Hurley**

COMPARATIVE STATICS FOR MARSHALLIAN DEMAND

When it comes to studying Marshallian demand, there are three types of questions economists are often interested in:

a) How will demand respond to a change in income?
b) How will demand respond to change in a goods own price?
c) How will the demand respond to a change in the price of another good?

Theoretically, these questions can be tackled using what economists refer to a comparative static analysis, which is just the application of the Implicit Function Theorem. Recall that the first order conditions for the UMP imply

$$\frac{\partial L}{\partial x_l} = \frac{\partial u(x^*)}{\partial x_l} - l^* p_l \leq 0, \qquad \frac{\partial L}{\partial x_l} x_l^* = 0, \text{ and} \qquad x_l^* \geq 0 \text{ for } l = 1,\ldots L, \text{ and}$$

$$\frac{\partial L}{\partial l} = w - p \cdot x^* \geq 0, \qquad \frac{\partial L}{\partial l} l^* = 0, \text{ and } l^* \geq 0.$$

If we are willing to assume the solution to these equation results in an interior solution, then we can write these conditions as

(1) $w - px^* = 0$ and

(2) $-\lambda^* p_l + \frac{\partial u(x^*)}{\partial x_l} = 0$ for $l = 1,\ldots L$.

Totally differentiating equations (1) and (2) with respect to w, p, l^*, and x^* yields

(3) $dw - \sum_{k=1}^{L} x_k^* dp_k - \sum_{k=1}^{L} p_k dx_k^* = 0$, and

(4) $-\lambda^* dp_l - p_l d\lambda^* + \sum_{k=1}^{L} \frac{\partial^2 u(x^*)}{\partial x_l \partial x_k} dx_k^* = 0$ for $l = 1,\ldots L$.

Now, if we want to figure out how a change in w affects Marshallian demand, we can set $dp_l = 0$ for all $l = 1,\ldots L$ and rewrite equations (3) and (4) in matrix notation:

(5) $$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} dw + \begin{bmatrix} 0 & -p_1 & \cdots & -p_L \\ -p_1 & \frac{\partial^2 u(x^*)}{\partial x_1^2} & \cdots & \frac{\partial^2 u(x^*)}{\partial x_1 \partial x_L} \\ \vdots & \vdots & \ddots & \vdots \\ -p_L & \frac{\partial^2 u(x^*)}{\partial x_L \partial x_1} & \cdots & \frac{\partial^2 u(x^*)}{\partial x_L^2} \end{bmatrix} \begin{bmatrix} d\lambda^* \\ dx_1^* \\ \vdots \\ dx_L^* \end{bmatrix} = 0.$$

Equation (5) can be solved for the comparative static effect of interest provided

(6) $$|J| = \begin{vmatrix} 0 & -p_1 & \cdots & -p_L \\ -p_1 & \frac{\partial^2 u(x^*)}{\partial {x_1}^2} & \cdots & \frac{\partial^2 u(x^*)}{\partial x_1 \partial x_L} \\ \vdots & \vdots & \ddots & \vdots \\ -p_L & \frac{\partial^2 u(x^*)}{\partial x_L \partial x_1} & \cdots & \frac{\partial^2 u(x^*)}{\partial {x_L}^2} \end{vmatrix} \neq 0.$$

This solution is

(7) $$\begin{bmatrix} \frac{d\lambda^*}{dw} \\ \frac{d{x_1}^*}{dw} \\ \vdots \\ \frac{d{x_L}^*}{dw} \end{bmatrix} = -\begin{bmatrix} 0 & -p_1 & \cdots & -p_L \\ -p_1 & \frac{\partial^2 u(x^*)}{\partial {x_1}^2} & \cdots & \frac{\partial^2 u(x^*)}{\partial x_1 \partial x_L} \\ \vdots & \vdots & \ddots & \vdots \\ -p_L & \frac{\partial^2 u(x^*)}{\partial x_L \partial x_1} & \cdots & \frac{\partial^2 u(x^*)}{\partial {x_L}^2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Alternatively, if we are interested in how a change in the price of p_1 affects Marshallian demand, we can set $dw = 0$ and $dp_l = 0$ for all $l = 2, \ldots L$ and rewrite equations (3) and (4) in matrix notation:

(8) $$\begin{bmatrix} -{x_1}^* \\ -\lambda^* \\ 0 \\ \vdots \\ 0 \end{bmatrix} dp_1 + \begin{bmatrix} 0 & -p_1 & \cdots & -p_L \\ -p_1 & \frac{\partial^2 u(x^*)}{\partial {x_1}^2} & \cdots & \frac{\partial^2 u(x^*)}{\partial x_1 \partial x_L} \\ \vdots & \vdots & \ddots & \vdots \\ -p_L & \frac{\partial^2 u(x^*)}{\partial x_L \partial x_1} & \cdots & \frac{\partial^2 u(x^*)}{\partial {x_L}^2} \end{bmatrix} \begin{bmatrix} d\lambda^* \\ d{x_1}^* \\ \vdots \\ d{x_L}^* \end{bmatrix} = 0.$$

Again, if $|J| \neq 0.$, then equation (8) can be solved for the comparative static effects of interest:

(9) $$\begin{bmatrix} \frac{d\lambda^*}{dp_1} \\ \frac{d{x_1}^*}{dp_1} \\ \vdots \\ \frac{d{x_L}^*}{dp_1} \end{bmatrix} = \begin{bmatrix} 0 & -p_1 & \cdots & -p_L \\ -p_1 & \frac{\partial^2 u(x^*)}{\partial {x_1}^2} & \cdots & \frac{\partial^2 u(x^*)}{\partial x_1 \partial x_L} \\ \vdots & \vdots & \ddots & \vdots \\ -p_L & \frac{\partial^2 u(x^*)}{\partial x_L \partial x_1} & \cdots & \frac{\partial^2 u(x^*)}{\partial {x_L}^2} \end{bmatrix}^{-1} \begin{bmatrix} {x_1}^* \\ \lambda^* \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The comparative static effects for prices p_2 through p_L will look similar to equation (9), with appropriate modifications to $\begin{bmatrix} \frac{d\lambda^*}{dp_1} \\ \frac{d{x_1}^*}{dp_1} \\ \vdots \\ \frac{d{x_L}^*}{dp_1} \end{bmatrix}$ and $\begin{bmatrix} {x_1}^* \\ \lambda^* \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. In general, these comparative static effects will be ambiguous, which has resulted in economists developing a variety of taxonomies (e.g. normal versus inferior goods, complements versus substitutes, and Giffen goods).