

# CLASSICAL CONSUMER PROBLEM

ECON 8001-2

Instructor: Terry Hurley

## INTRODUCTION

The last couple of lectures defined various types of preference relations and discussed how assuming that a consumer's preference relation could be characterized as such allows us to generate a utility function with certain classical characteristics. We will now take advantage of this idea by starting with a consumer's utility function and using it to characterize choice in certain environments. Specifically, we will be interested in how consumers choose between bundles of *commodities* offered by a *competitive market economy*. This competitive market economy will be defined as an institution where consumers are endowed with commodities that they can voluntarily exchange with other consumers at known exchange rates, referred to as *prices*. These prices are taken as exogenous and tell us something about a consumer's *wealth* when combined with its commodity endowment. Ultimately, it is this wealth and prices that define the consumer's feasible choice set.

## THE SETUP

Some More Notation:

- $p \in \mathfrak{R}^L$ : Vector of commodity prices.
- $p_l \in \mathfrak{R}$ : Specific price of commodity  $l$  for  $l = 1, 2, \dots, L$ . Note that positive values represent what a consumer pays in order to receive an additional unit of a commodity, while negative values represent what a consumer receives in order to accept an additional unit of a commodity.
- $x^e \in X$ : Vector of commodity endowments. That is, what a consumer has available to exchange.
- $x_l^e \geq 0$ : Specific quantity of commodity  $l$  endowment for  $l = 1, 2, \dots, L$ .
- $w \geq 0$ : Consumer wealth or the value of the consumer's commodity endowment:  $w = p \cdot x^e$ .
- $u: X \rightarrow \mathfrak{R}$ : Utility function generated by a consumer's rational preference relation over  $X$ .

To keep our discussion manageable, we will focus on  $p \gg 0$  (meaning  $\{p \in \mathfrak{R}^L: p_l > 0 \text{ for } l = 1, 2, \dots, L\}$ ). To keep it interesting, we also assume  $w > 0$  so the feasible choice set includes more than the origin.

Definition:

*Budget Set* (denoted by  $B_{p,w}$ ): The set of all affordable commodity bundles for a consumer who faces market prices  $p$  with wealth  $w$ :  $B_{p,w} \equiv \{x \in \mathfrak{R}_+^L: w \geq p \cdot x\}$ .

This *Budget Set* is referred to in MWG as the *Walrasian Budget Set* and has been called by other names. It should be noted that this budget set is about as simple as it gets. There are many other more complicated, but also completely realistic budget sets one could explore (e.g. two-tier pricing and quotas).

Question: Given the opportunity to participate in this market economy, what will the rational consumer with utility function  $u(\cdot)$  do?

## CLASSICAL CONSUMER PROBLEM

ECON 8001-2

Instructor: Terry Hurley

To answer this question, note that the consumer's utility function can be used to compare commodity bundles in the consumer's budget set and to determine which bundles are the most preferred:

$$\text{CP1} \quad x(p, w) = \{x \in B_{p,w}: u(x) \geq u(y) \text{ for all } y \in B_{p,w}\}.$$

But, another way to write **CP1** is

$$\text{CP1}' \quad \max_{x \geq 0} u(x) \text{ such that } w \geq p \cdot x.$$

This is the classic utility maximization problem of the consumer that is referred to as the *UMP* by MWG.

Given this problem, the first question that may come to mind is: How do we know  $x(p, w) \neq \emptyset$ ? Alternatively, if we know the conditions under which  $x(p, w) \neq \emptyset$ , can we also find the conditions under which the  $x(p, w)$  contains a single vector?

Answering these questions gets quite technical and is of marginal value when it comes to doing more empirical work. So for now, let us just state some useful propositions:

**PROPOSITION CP1:** If  $f: \mathfrak{R}^N \rightarrow \mathfrak{R}$  is a continuous function,  $A \subset \mathfrak{R}^N$ , and  $A$  is compact (i.e. closed and bounded), then there exists  $x \in A$  such that  $f(x) \geq f(y)$  for all  $y \in A$ .

This more general proposition leads immediately to:

**COROLARY CP1:** If  $p \gg 0$  and  $u(\cdot)$  is continuous, then the classic utility maximization problem has a solution.

This is true because the utility function is continuous and  $B_{p,w}$  is a compact set. It means that if a consumer has a preference relation that generates a continuous utility function defined over commodities traded in a competitive market economy with strictly positive prices (What possible problem can emerge if some prices are 0?), then there will be some optimal vector or vectors of commodities that maximize this consumer's utility or are most preferred by the consumer given its feasible alternatives. We know that rationality is necessary for a preference relation to generate a utility function. What about continuity?

**PROPOSITION CP2:** If there is a rational preference relation  $\underline{f}$  on  $X$  that is continuous, then there is a continuous utility function  $u: X \rightarrow \mathfrak{R}$  that represents  $\underline{f}$ .

Now, if we in fact believe consumers choose preferred commodity bundles, this is precisely the answer to our first question, which MWG refer to as the *Walrasian Demand Correspondence*, but has also been referred to as the *Market Demand Correspondence/Function*, *Marshallian Demand Correspondence/Function*, etc. We will refer to it as *Marshallian Demand*. The most important thing to note for distinguishing this Marshallian Demand relation from other demand relations we will define later is that it depends on commodity prices and wealth.

## CLASSICAL CONSUMER PROBLEM

ECON 8001-2

Instructor: Terry Hurley

Given our current assumptions, it is straightforward to prove some important properties of utility functions and Marshallian Demand:

**PROPOSITION CP3:** If  $u(\cdot)$  is a utility function that represents a continuous, rational preference relation  $\underline{f}$  on  $X$  and  $f: \mathfrak{R} \rightarrow \mathfrak{R}$  is a strictly increasing function such that  $f(y) > f(z)$  for all  $y, z \in \mathfrak{R}$  if and only if  $y > z$ , then the function  $v(x) \equiv f(u(x))$  is also a utility function that represents the continuous, rational preference relation  $\underline{f}$  on  $X$ .

**Proof:** Since  $u(\cdot)$  is utility function for the preference relation  $\underline{f}$  on  $X$ , for all  $x, y \in X$ ,  $u(x) \geq u(y)$  if and only if  $x \underline{f} y$ . If  $v(\cdot)$  is also a utility function for the preference relation  $\underline{f}$  on  $X$ , for all  $x, y \in X$ ,  $v(x) \geq v(y)$  if and only if  $x \underline{f} y$ . Let us assume to the contrary that there exists some  $x, y \in X$  where  $v(y) > v(x)$  for  $x \underline{f} y$ . By the definition of a strictly increasing function,  $v(y) > v(x)$  if and only if  $u(y) > u(x)$ . However, if  $u(y) > u(x)$ , not  $u(x) \geq u(y)$  or not  $x \underline{f} y$ , a contradiction.

Q.E.D

The importance of PROPOSITION CP3 is that it tells us a utility function is not unique, which means we should not read too much into the numbers it generates. If  $u(y) = 86$  and  $u(z) = 43$ , we can say  $y$  is preferred to  $z$ , but we cannot say  $y$  is twice as good as  $z$ . A simple example illustrates the point. Note that if  $f(x) = x^2$ , a strictly increasing function,  $v(y) = 7,396$  and  $v(z) = 1,849$  is the same as  $u(y) = 86$  and  $u(z) = 43$  by PROPOSITION CP3, but  $7,396$  is far more than twice  $1,849$ . What implication might this have for normative analysis?

**PROPOSITION CP4:** If there is a continuous and rational preference relation  $\underline{f}$  on  $X$ , then the Marshallian Demand is homogeneous of degree 0 in prices and wealth:  $x(p, w) = x(ap, aw)$  for any  $a > 0$ .

**Proof:** By definition,  $f: x \rightarrow \mathfrak{R}$  for  $x \in \mathfrak{R}^N_+$ ,  $r = \dots, -1, 0, 1, \dots$ , and  $N = 1, 2, \dots$  is homogeneous of degree  $r$  if  $f(ax) = a^r f(x)$  for all  $a > 0$ . Note that  $x(p, w) \equiv \{x \in B_{p,w}: u(x) \geq u(y) \text{ for all } y \in B_{p,w}\}$  and  $x(ap, aw) \equiv \{x \in B_{ap,aw}: u(x) \geq u(y) \text{ for all } y \in B_{ap,aw}\}$ .  $B_{ap,aw} \equiv \{x \in \mathfrak{R}^L_+: aw \geq ap \cdot x\} = \{x \in \mathfrak{R}^L_+: w \geq p \cdot x\} \equiv B_{p,w}$ . Therefore,  $x(ap, aw) \equiv \{x \in B_{p,w}: u(x) \geq u(y) \text{ for all } y \in B_{p,w}\} \equiv x(p, w)$ .

Q.E.D.

The implication of PROPOSITION CP4 is that prices and wealth are relative or that doubling prices has no effect on a consumer's preferred commodity vectors if we also double wealth.

Now, if we are willing to assume the preference relation is locally nonsatiated, we can say a little more about Marshallian Demand:

**PROPOSITION CP5:** Suppose  $u(\cdot)$  is a continuous utility function representing a locally nonsatiated, continuous, and rational preference relation  $\underline{f}$  on  $X = \mathfrak{R}^L_+$ . Then the Marshallian Demand  $x(p, w)$  satisfies *Walras Law*:  $p \cdot x = w$  for all  $x \in x(p, w)$ .

## CLASSICAL CONSUMER PROBLEM

ECON 8001-2

Instructor: Terry Hurley

**Proof:** Assume instead that  $w > p \cdot x$  for some  $x \in x(p, w)$ . Then by definition  $x \in B_{p,w}$  and  $u(x) \geq u(y)$  for all  $y \in B_{p,w}$ , which means  $x \underline{f} y$  for all  $y \in B_{p,w}$ . Define the set  $W(z) = \{z' : \|z - x\| \geq \|z' - x\| \text{ for all } z' \in X\}$ . Choose a  $z$  such that  $w \geq p \cdot z > p \cdot x$  and  $W(z) \subset B_{p,w}$ . By local nonsatiation, for  $\epsilon = \|z - x\|$  there is a  $y \in W(z)$  such that  $y \mathbf{f} x$ , which contradicts  $x \underline{f} y$  for all  $y \in B_{p,w}$ . Q.E.D.

PROPOSITION CP5 is important because it tells us that consumers will fully exhaust their budget if they have locally nonsatiated preferences.

Finally, let us consider what adding convexity gets us:

**PROPOSITION CP6:** Suppose  $u(\cdot)$  is a continuous utility function representing a convex (strictly convex), locally nonsatiated, continuous, and rational preference relation  $\underline{f}$  on  $X = \mathfrak{R}^L_+$ . Then the Marshallian Demand  $x(p, w)$  is a convex (singleton) set.

**Proof:** We will start with the case of convexity. Let  $x, x' \in x(p, w)$ . By the definition of Marshallian Demand  $u(x) \geq u(x')$  and  $u(x') \geq u(x)$ , which implies  $x \underline{f} x'$  and  $x' \underline{f} x$ .

PROPOSITION CP5 also implies  $p \cdot x = p \cdot x' = w$ . Now, for any  $a \in [0, 1]$ , let  $x'' = ax + (1 - a)x'$ , which implies  $p \cdot x'' = w$ , so  $x'' \in B_{p,w}$ . Note that  $x \underline{f} x$  by Proposition IC1 (b). Therefore, by convexity,  $x \underline{f} x$  and  $x' \underline{f} x$ , implies  $ax + (1 - a)x' \underline{f} x$  or  $x'' \underline{f} x$ . Since  $x'' \underline{f} x$ ,  $x'' \in B_{p,w}$ , and  $x \underline{f} y$  for all  $y \in B_{p,w}$ , transitivity implies  $x'' \underline{f} y$  for all  $y \in B_{p,w}$  or  $x'' \in x(p, w)$ .

Now we can turn to strict convexity. Suppose there exists  $x, x' \in x(p, w)$  where  $x \neq x'$ , which is true if  $x(p, w)$  is not a singleton set. By the definition of a Marshallian Demand correspondence  $u(x) \geq u(x')$  and  $u(x') \geq u(x)$ , which implies  $x \underline{f} x'$  and  $x' \underline{f} x$ . PROPOSITION CP5 also implies  $p \cdot x = p \cdot x' = w$ . Now, for any  $a \in (0, 1)$ , let  $x'' = ax + (1 - a)x'$ , which implies  $p \cdot x'' = w$ , so  $x'' \in B_{p,w}$ . Note that  $x \underline{f} x$  by PROPOSITION IC1 b). Therefore, by strict convexity,  $x \underline{f} x$  and  $x' \underline{f} x$ , implies  $ax + (1 - a)x' \mathbf{f} x$  or  $x'' \mathbf{f} x$ . Now  $x'' \mathbf{f} x$  and  $x'' \in B_{p,w}$  contradicts  $x \underline{f} y$  for all  $y \in B_{p,w}$ . Since  $x \sim x'$ , transitivity from PROPOSITION IC1 (b) implies  $x' \underline{f} y$  for all  $y \in B_{p,w}$  is also contradicted. Q.E.D.

PROPOSITION CP6 is particularly useful because it tells us conditions under which we can be confident that the Marshallian Demand correspondence will be a singleton set. PROPOSITION CP6 can also be proven by first showing convexity (strict convexity) yields utility functions that are quasiconcave (strictly quasiconcave) for a continuous, rational preference relation:

**COROLLARY CP6:** Suppose  $u(\cdot)$  is a continuous utility function representing a convex (strictly convex), locally nonsatiated, continuous, and rational preference relation  $\underline{f}$  on  $X = \mathfrak{R}^L_+$ , then  $u(\cdot)$  is quasiconcave (strictly quasiconcave).

Knowing that the utility function is quasiconcave or strictly quasiconcave becomes very helpful when evaluating second-order conditions for an optimum.

## CLASSICAL CONSUMER PROBLEM

ECON 8001-2

Instructor: Terry Hurley

Now we might want to complain about the restrictiveness of these assumptions. We wouldn't be alone, which has resulted in a technically advanced literature that explores how relaxing them affects these properties.

We will now add one more assumption to facilitate further analysis using calculus. In particular, we will assume that  $u(\cdot)$  is twice continuously differentiable. With this additional assumption we can set up **CP1'** as a constrained optimization problem and derive the Kuhn-Tucker necessary and sufficient conditions for finding Marshallian Demand. The Lagrangian for this problem is written as

$$\mathbf{CP3} \quad L = u(x) + I(w - p \cdot x)$$

The first-order necessary conditions for this problem are

$$\mathbf{CP4} \quad \frac{\partial L}{\partial x_l} = \frac{\partial u(x^*)}{\partial x_l} - I^* p_l \leq 0, \quad \frac{\partial L}{\partial x_l} x_l^* = 0, \text{ and } x_l^* \geq 0 \text{ for } l = 1, \dots, L, \text{ and}$$

$$\mathbf{CP5} \quad \frac{\partial L}{\partial I} = w - p \cdot x^* \geq 0, \quad \frac{\partial L}{\partial I} I^* = 0, \text{ and } I^* \geq 0.$$

If  $\nabla u(x) = \left[ \frac{\partial u(x)}{\partial x_1}, \dots, \frac{\partial u(x)}{\partial x_L} \right]$ , CP4 can also be written in matrix notation as

$$\mathbf{CP4'} \quad \nabla u(x^*) - I^* p \leq 0, \quad x^* \cdot [\nabla u(x^*) - I^* p] = 0, \text{ and } x^* \geq 0.$$

Note that the *Constraint Qualification* for the Kuhn-Tucker problem is satisfied here because the inequality constraint is a convex set of linear functions.

One set of sufficiency conditions for this Kuhn-Tucker problem to yield a global maximum are:

- (i) the objective function,  $u(x)$ , is differentiable and quasiconcave in the nonnegative orthant,
- (ii) the constraint function,  $p \cdot x$ , is differentiable and quasiconvex in the nonnegative orthant,
- (iii)  $x^*$  satisfies the first order necessary condition in equations CP4 and CP5, and
- (iv)  $\frac{\partial u(x^*)}{\partial x_l} < 0$  for at least one variable  $x_l$ ;  $\frac{\partial u(x^*)}{\partial x_l} \geq 0$  for some variables that can take on a positive value without violating the constraints;  $u(x)$  is twice continuously differentiable and  $\nabla u(x) \neq 0$ ; **or**  $u(\cdot)$  is concave.

With our current list of restriction, (i) is satisfied by COROLLARY CP6; (ii) is satisfied because a convex set is also quasiconvex and I have already told you the budget set is convex (though did not prove it); and (iii) tells us which point to pick. (iv) is a little tougher. Note that PROPOSITION CP5 tells us Walras law is in effect, which when combined with CP5 suggests

$$I^* > 0. \text{ For } I^* > 0, \frac{\partial u(x^*)}{\partial x_l} = I^* p_l > 0 \text{ meaning } \frac{\partial u(x^*)}{\partial x_l} > 0 \text{ for all } l = 1, \dots, L \text{ where } x_l^* > 0.$$

That such an  $x_l^*$  must exist is assured because we assumed  $w > 0$  and Walras Law cannot be satisfied for  $w > 0$  unless some  $x_l > 0$ . Note however that if we choose to relax some of our

## CLASSICAL CONSUMER PROBLEM

ECON 8001-2

Instructor: Terry Hurley

current assumptions, we must do so in a way that satisfies (i) – (iv) or we cannot be sure the Kuhn-Tucker conditions in CP4 and CP5 will lead us to  $x(p, w)$ .

Returning to CP4 and CP5, with the assumptions of a twice continuously differentiable utility function representing a strictly convex, locally nonsatiated, continuous, and rational preference relation  $\tilde{f}$  on  $X$  (which I will no longer write out and we will assume from here on out unless otherwise noted), the Kuhn-Tucker problem will have a unique solution  $x^* = x(p, w)$ , our Marshallian Demand, and  $I^* = I(p, w)$  that will be of great interest to us (note that we could allow for multiple solutions by relaxing strict convexity, but this can get a bit tedious).

Now that we have our Marshallian Demand and have proved that it must satisfy a variety of interesting properties, several questions come to mind.

Questions: What will happen to Marshallian Demand when prices change?  
What will happen when income changes?

Answering these questions is what is referred to as a comparative static analysis. This comparative static analysis turns out to be not so illuminating in one sense, but quite illuminating in another. While we are used to seeing the demand for a commodity fall as its price increases,  $\frac{\partial x_l(p, w)}{\partial p_l} < 0$ , there are examples of when the demand for a commodity has increased when its

price increased,  $\frac{\partial x_l(p, w)}{\partial p_l} > 0$ . This type of commodity has been given the name *Giffen Good*.

None of our assumptions above rule this out.

If we cannot say much about own price effects, you can guess that we cannot say much about cross-price effects either:  $\frac{\partial x_l(p, w)}{\partial p_k} > / = / < 0$  for  $l \neq k$ . The sign of this cross-price effect has

instead been used to create the taxonomy of *Marshallian Complements*  $\left( \frac{\partial x_l(p, w)}{\partial p_k} < 0 \right)$  and

*Marshallian Substitutes*  $\left( \frac{\partial x_l(p, w)}{\partial p_k} > 0 \right)$ .

We cannot even say for certain how the demand for a good will respond to increases in wealth:  $\frac{\partial x_l(p, w)}{\partial w} > / = / < 0$ , which has led to the taxonomy of *Normal Goods*  $\left( \frac{\partial x_l(p, w)}{\partial w} > 0 \right)$  and

*Inferior Goods*  $\left( \frac{\partial x_l(p, w)}{\partial w} < 0 \right)$ .

While we cannot say much about how the Marshallian Demand for a particular commodity responds to changes in prices or wealth, the fact that Marshallian Demand is homogeneous in prices and wealth does provide us with an interesting relationship.

## CLASSICAL CONSUMER PROBLEM

ECON 8001-2

Instructor: Terry Hurley

**PROPOSITION CP7:** If the Marshallian Demand,  $x(p, w)$ , is differentiable in prices and wealth, then for all  $p, w$ , and  $l = 1, \dots, L$ , 
$$\sum_{k=1}^L \frac{\partial x_l(p, w)}{\partial p_k} p_k + \frac{\partial x_l(p, w)}{\partial w} w = 0.$$

The result can also be written in elasticity form. The cross/cross price elasticity of Marshallian Demand is defined as  $e_{lk}(p, w) = \frac{\partial x_l(p, w)}{\partial p_k} \frac{p_k}{x_l(p, w)}$ , while the income elasticity is defined as

$e_{lw}(p, w) = \frac{\partial x_l(p, w)}{\partial w} \frac{w}{x_l(p, w)}$ . With these, PROPOSITION CP7 then implies

$$\sum_{k=1}^L e_{lk}(p, w) + e_{lw}(p, w) = 0$$
, which says that the sum of the price, cross-price, and income elasticities for a commodity must equal 0.

When Marshallian Demand satisfies Walras Law, we have two more interesting properties:

**PROPOSITION CP8:** If the Marshallian Demand,  $x(p, w)$ , is differentiable in prices and wealth and satisfies Walras Law, then for all  $p$  and  $w$

- (i) 
$$\sum_{l=1}^L \frac{\partial x_l(p, w)}{\partial p_k} p_l + x_k(p, w) = 0$$
 for all  $k = 1, \dots, L$  and
- (ii) 
$$\sum_{l=1}^L \frac{\partial x_l(p, w)}{\partial w} p_l = 1.$$

Note that the budget share of a commodity is defined as  $b_l(p, w) = p_l x_l(p, w) / w$ . With this and the elasticity definitions, (i) becomes 
$$\sum_{l=1}^L b_l(p, w) e_{lk}(p, w) + b_k(p, w) = 0$$
, while (ii) becomes

$$\sum_{l=1}^L b_l(p, w) e_{lw}(p, w) = 1.$$

The properties in PROPOSITION CP7 and CP8 are used extensively in empirical work on consumer demand. By imposing these restrictions when estimating a Marshallian Demand system, more efficient parameter estimates can be obtained. Also, the conditions provide an opportunity to test the theory by estimating a Marshallian Demand system and checking if these restrictions are indeed satisfied.

Another quantity that has been of some interest to economists is the *Indirect Utility Function*:  $v(p, w) = u(x(p, w))$ . The Indirect Utility Function has a number of useful properties that will come in handy both theoretically and empirically.

**PROPOSITION CP9:** Suppose that  $u(\cdot)$  is a continuous utility function representing a locally nonsatiated, rational preference relation  $\underline{f}$  on  $X = \Re^L_+$ . Then the Indirect Utility Function  $v(p, w) = u(x(p, w))$  is

## CLASSICAL CONSUMER PROBLEM

ECON 8001-2

Instructor: Terry Hurley

- (i) homogeneous of degree 0 in  $p$  and  $w$ ;
- (ii) strictly increasing in  $w$  and nonincreasing in  $p$ ;
- (iii) quasiconvex in  $p$ :  $v(ap + (1 - a)p', w) \leq v(p', w)$  for  $v(p', w) \geq v(p, w)$ , and  $a \in (0, 1)$ ; and
- (iv) continuous in  $p$  and  $w$ .

**Proof:** We will prove (i), (ii), and (iii) and leave the more advanced proof of (iv) to those of you who get excited about continuity proofs (not me!).

The proof of (i) is terribly straightforward. If  $v(p, w)$  is homogenous of degree 0 in  $p$  and  $w$ , then  $v(p, w) = v(ap, aw)$  for any  $a > 0$ . By definition,  $v(ap, aw) = u(x(ap, aw))$ . By PROPOSITION CP4,  $x(p, w) = x(ap, aw)$ . Therefore,  $v(ap, aw) = u(x(p, w)) = v(p, w)$ .

The proof of (ii) gets a bit more involved. For  $v(p, w)$  to be strictly increasing in  $w$ ,  $v(p, w') > v(p, w)$  or  $u(x(p, w')) > u(x(p, w))$  for  $w' > w$ . By definition  $B_{p,w'} \equiv \{x \in \mathfrak{R}^L_+ : w' \geq p \cdot x\}$  and  $B_{p,w} \equiv \{x \in \mathfrak{R}^L_+ : w \geq p \cdot x\}$ , which implies  $B_{p,w} \subset B_{p,w'}$ . PROPOSITION CP5 implies  $w' = p \cdot x(p, w')$  and  $w = p \cdot x(p, w)$ , so  $p \cdot x(p, w') > p \cdot x(p, w)$  and  $x(p, w') \neq x(p, w)$ . By definition,  $x(p, w') = \{x \in B_{p,w'} : u(x) \geq u(y) \text{ for all } y \in B_{p,w'}\}$  or  $x(p, w') \preceq x(p, w)$  and  $v(p, w') \geq v(p, w)$ . Since  $w' > p \cdot x(p, w)$ , PROPOSITION CP5 also implies not  $x(p, w) \preceq x(p, w')$  meaning not  $v(p, w) \geq v(p, w')$ . Therefore,  $v(p, w') > v(p, w)$  as required.

For  $v(p, w)$  to be nonincreasing in  $p$ ,  $v(p', w) \geq v(p, w)$  meaning  $u(x(p', w)) \geq u(x(p, w))$  for  $p \geq p'$  and  $p \neq p'$ . By definition  $B_{p',w} \equiv \{x \in \mathfrak{R}^L_+ : w \geq p' \cdot x\}$  and  $B_{p,w} \equiv \{x \in \mathfrak{R}^L_+ : w \geq p \cdot x\}$ , which implies  $B_{p,w} \subset B_{p',w}$ . By definition,  $x(p', w) = \{x \in B_{p',w} : u(x) \geq u(y) \text{ for all } y \in B_{p',w}\}$  or  $x(p', w) \preceq x(p, w)$  and  $v(p', w) \geq v(p, w)$ .

The proof of (iii) gets really involved. We are given  $v(p', w) \geq v(p, w)$ , or  $x(p', w) \preceq x(p, w)$ , and  $a \in (0, 1)$ . We want to show  $v(ap + (1 - a)p', w) \leq v(p', w)$  or  $x(p', w) \preceq x(ap + (1 - a)p', w)$ . Notice that if we can show  $x(ap + (1 - a)p', w) \in B_{p,w}$  then we can say  $x(p', w) \preceq x(p, w) \preceq x(ap + (1 - a)p', w)$  by transitivity. Alternatively, if we can show  $x(ap + (1 - a)p', w) \in B_{p',w}$ , then we can say  $x(p', w) \preceq x(ap + (1 - a)p', w)$  directly. Therefore, if we can show that  $x(ap + (1 - a)p', w) \in B_{p,w}$  or  $x(ap + (1 - a)p', w) \in B_{p',w}$  we are done.

Let us suppose not. That is,  $x(ap + (1 - a)p', w) \notin B_{p,w}$  and  $x(ap + (1 - a)p', w) \notin B_{p',w}$  or that there is an  $x \in \{x : w \geq (ap + (1 - a)p') \cdot x\}$  such that  $x \notin \{x : w \geq p \cdot x\}$  and  $x \notin \{x : w \geq p' \cdot x\}$ . Note that  $x \in \{x : w \geq (ap + (1 - a)p') \cdot x\}$  implies  $w \geq (ap + (1 - a)p') \cdot x$ ;  $x \notin \{x : w \geq p \cdot x\}$  implies  $w < p \cdot x$ ; and  $x \notin \{x : w \geq p' \cdot x\}$  implies  $w < p' \cdot x$ . Note also that  $w < p \cdot x$  and  $w < p' \cdot x$  imply  $aw < ap \cdot x$  and  $(1 - a)w < (1 - a)p' \cdot x$ . Summing then yields  $aw + (1 - a)w < ap \cdot x + (1 - a)p' \cdot x$  or  $w < ap \cdot x + (1 - a)p' \cdot x$ , which is a contradiction. Q.E.D.

We are about ready to move on from the classic utility maximization problem, but before we do an illustration that all this really works would be reassuring. Suppose our consumer must choose between two different commodities:  $x_1$  and  $x_2$ . The utility function that represents its preferences is  $u(x_1, x_2) = x_1^{0.75} x_2^{0.5}$ . The prices the consumer faces are  $p_1$  and  $p_2$ , while wealth is  $w$ . The

## CLASSICAL CONSUMER PROBLEM

ECON 8001-2

Instructor: Terry Hurley

Lagrangian for this problem is  $L = x_1^{0.75} x_2^{0.5} + \lambda(w - p_1 x_1 - p_2 x_2)$ . The first-order conditions are

$$\text{CP6} \quad \frac{\partial L}{\partial x_1} = 0.75 x_1^{-0.25} x_2^{0.5} - I^* p_1 \leq 0, \quad \frac{\partial L}{\partial x_1} x_1^* = 0, \quad \text{and} \quad x_1^* \geq 0,$$

$$\text{CP7} \quad \frac{\partial L}{\partial x_2} = 0.5 x_1^{0.75} x_2^{-0.5} - I^* p_2 \leq 0, \quad \frac{\partial L}{\partial x_2} x_2^* = 0, \quad \text{and} \quad x_2^* \geq 0,$$

$$\text{CP8} \quad \frac{\partial L}{\partial I} = w - p_1 x_1^* - p_2 x_2^* \geq 0, \quad \frac{\partial L}{\partial I} I^* = 0, \quad \text{and} \quad I^* \geq 0.$$

Note that there are eight different types of solutions for this problem:

- (i)  $x_1^* > 0, x_2^* > 0, \text{ and } I^* > 0,$
- (ii)  $x_1^* > 0, x_2^* > 0, \text{ and } I^* = 0,$
- (iii)  $x_1^* > 0, x_2^* = 0, \text{ and } I^* > 0,$
- (iv)  $x_1^* > 0, x_2^* = 0, \text{ and } I^* = 0,$
- (v)  $x_1^* = 0, x_2^* > 0, \text{ and } I^* > 0,$
- (vi)  $x_1^* = 0, x_2^* > 0, \text{ and } I^* = 0,$
- (vii)  $x_1^* = 0, x_2^* = 0, \text{ and } I^* > 0, \text{ and}$
- (viii)  $x_1^* = 0, x_2^* = 0, \text{ and } I^* = 0.$

Therefore, a complete treatment of the problem would carefully rule each one out and also check the second-order conditions. In the interest of time, I will focus on interior solutions and assert that the second-order conditions hold, so we know we have a global maximum. For an interior solution,  $x_1^* > 0, x_2^* > 0, \text{ and } I^* > 0$ . Therefore,

$$\text{CP6}' \quad \frac{\partial u(x_1^*, x_2^*)}{\partial x_1} = 0.75 x_1^{-0.25} x_2^{0.5} = I^* p_1, \quad \text{and}$$

$$\text{CP7}' \quad \frac{\partial u(x_1^*, x_2^*)}{\partial x_2} = 0.5 x_1^{0.75} x_2^{-0.5} = I^* p_2.$$

Taking the ratio of these two equations yields

$$\begin{aligned} \text{CP9} \quad \frac{\frac{\partial u(x_1^*, x_2^*)}{\partial x_1}}{\frac{\partial u(x_1^*, x_2^*)}{\partial x_2}} &= \frac{0.75 x_1^{-0.25} x_2^{0.5}}{0.5 x_1^{0.75} x_2^{-0.5}} = \frac{I^* p_1}{I^* p_2} \\ &= \frac{3 x_2^*}{2 x_1^*} = \frac{p_1}{p_2}. \end{aligned}$$

Note that in general for the utility maximization problem in CP1' where  $u(\cdot)$  is differentiable,

$$\text{CP9}' \quad \frac{\frac{\partial u(x^*)}{\partial x_1}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k}$$

## CLASSICAL CONSUMER PROBLEM

ECON 8001-2

Instructor: Terry Hurley

if  $x_l^* > 0$  and  $x_k^* > 0$ , which says that at the maximum, the *Marginal Rate of Substitution* between two goods will equal their ratio of prices. Now  $I^* > 0$  implies

**CP10**  $w = p_1 x_1^* + p_2 x_2^*.$

Equation CP9 implies  $x_2^* = \frac{2}{3} \frac{p_1}{p_2} x_1^*$ . Substituting this into CP10, we can then solve for

$x_1^* = \frac{3w}{5p_1}$ . Substituting again,  $x_2^* = \frac{2w}{5p_2}$ . To find  $I^*$ , we can substitute  $x_1^*$  and  $x_2^*$  into CP6'

and solve:  $I^* = 0.75 \left(\frac{3}{5}\right)^{-0.25} \left(\frac{2}{5}\right)^{0.5} \left(\frac{1}{p_1}\right)^{0.75} \left(\frac{1}{p_2}\right)^{0.5} w^{0.25} = \frac{5}{4} \left(\frac{3}{5}\right)^{0.75} \left(\frac{2}{5}\right)^{0.5} \left(\frac{1}{p_1}\right)^{0.75} \left(\frac{1}{p_2}\right)^{0.5} w^{0.25}.$

Substituting these Marshallian Demands into the utility function yields the indirect utility function:

**CP11**  $v(p_1, p_2, w) = \left(\frac{3w}{5p_1}\right)^{0.75} \left(\frac{2w}{5p_2}\right)^{0.5} = \left(\frac{3}{5p_1}\right)^{0.75} \left(\frac{2}{5p_2}\right)^{0.5} w^{1.25}.$

It should be pretty clear that if we multiply  $p_1$ ,  $p_2$ , and  $w$  by any  $a > 0$ , we will not change  $x_1^*$ ,  $x_2^*$  and  $v(p_1, p_2, w)$ , so they are all homogeneous of degree 0 in  $p_1$ ,  $p_2$ , and  $w$  (which agrees with PROPOSITION CP4 and PROPOSITION CP9(i)).

If we differentiate  $v(p_1, p_2, w)$  with respect to  $w$ , we get  $\frac{\partial v(p_1, p_2, w)}{\partial w} =$

$\frac{5}{4} \left(\frac{3}{5}\right)^{0.75} \left(\frac{2}{5}\right)^{0.5} \left(\frac{1}{p_1}\right)^{0.75} \left(\frac{1}{p_2}\right)^{0.5} w^{0.25} > 0$ , so the indirect utility function is strictly increasing in

wealth (in agreement with PROPOSITION CP9(ii)). It is also interesting to note that

$\frac{\partial v(p_1, p_2, w)}{\partial w} = I^*$ . This is not a coincidence! The property will hold in general (i.e.  $\frac{\partial v(p, w)}{\partial w} =$

$I^*$  for problem in CP1').

If we differentiate  $v(p_1, p_2, w)$  with respect to  $p_1$  and  $p_2$ , we get

$\frac{\partial v(p_1, p_2, w)}{\partial p_1} = -0.75 \left(\frac{3}{5p_1}\right)^{0.75} \left(\frac{1}{p_1}\right)^{1.75} \left(\frac{2}{5p_2}\right)^{0.5} w^{1.25} < 0$  and

$\frac{\partial v(p_1, p_2, w)}{\partial p_2} = -0.5 \left(\frac{3}{5p_1}\right)^{0.75} \left(\frac{2}{5p_2}\right)^{0.5} \left(\frac{1}{p_2}\right)^{1.5} w^{1.25} < 0$ , so the indirect utility function is

nonincreasing in prices (also in agreement with PROPOSITION CP8(ii)).

If we differentiate  $v(p_1, p_2, w)$  twice with respect to  $p_1$  and  $p_2$ , we get the Hessian matrix with respect to  $p$ :

**CLASSICAL CONSUMER PROBLEM**

**ECON 8001-2**

**Instructor: Terry Hurley**

**CP12**

$$[H_p] = \begin{bmatrix} \frac{\partial^2 v(p_1, p_2, w)}{\partial p_1^2} & \frac{\partial^2 v(p_1, p_2, w)}{\partial p_1 \partial p_2} \\ \frac{\partial^2 v(p_1, p_2, w)}{\partial p_1 \partial p_2} & \frac{\partial^2 v(p_1, p_2, w)}{\partial p_2^2} \end{bmatrix} =$$

$$\begin{bmatrix} 1.75 \times 0.75 \times \left(\frac{1}{p_1}\right)^{2.75} \left(\frac{1}{p_2}\right)^{0.5} & 0.75 \times 0.5 \times \left(\frac{1}{p_1}\right)^{1.75} \left(\frac{1}{p_2}\right)^{1.5} \\ 0.75 \times 0.5 \times \left(\frac{1}{p_1}\right)^{1.75} \left(\frac{1}{p_2}\right)^{1.5} & 1.5 \times 0.5 \times \left(\frac{1}{p_1}\right)^{0.75} \left(\frac{1}{p_2}\right)^{2.5} \end{bmatrix} \left(\frac{3}{5}\right)^{0.75} \left(\frac{2}{5}\right)^{0.5} w^{1.25}.$$

Recall that a function is convex if and only if its Hessian matrix is everywhere positive semidefinite. For  $[H_p]$ , this implies  $\frac{\partial^2 v(p_1, p_2, w)}{\partial p_1^2} > 0$  and  $|H_p| > 0$ . For any  $p_1 > 0$  and  $p_2 > 0$ ,

$\frac{\partial^2 v(p_1, p_2, w)}{\partial p_1^2} > 0$ . This is also true for  $|H_p|$ :

$$|H_p| = \left( 0.5 \times 0.75 \times \left(\frac{1}{p_1}\right)^{3.5} \left(\frac{1}{p_2}\right)^3 (1.75 \times 1.5 - 0.75 \times 0.5) \right) \left(\frac{3}{5}\right)^{0.75} \left(\frac{2}{5}\right)^{0.5} w^{1.25} > 0,$$

which allows us to conclude that  $v(p_1, p_2, w)$  is convex in  $p_1$  and  $p_2$ . But we also know that any convex function is quasiconvex, so we are in agreement with PROPOSITION CP9(iii).

Finally, inspection of  $v(p_1, p_2, w)$  should be enough to convince you it is continuous in  $p_1, p_2$ , and  $w$ , in agreement with PROPOSITION CP9(iv).