**Arrow-Enthoven Sufficiency Theorem**
In Chiang, A.C. (1984).
Fundamental Methods Of Mathematical Economics.
McGraw-Hill Inc., New York
pp. 744-746

Given the nonlinear program for $x \in \mathbb{R}_+^n$

$$\max_{x \geq 0} f(x) \text{ subject to } g^i(x) \leq r_i \text{ for } i = 1,\ldots,m$$

If the following conditions are satisfied:

(a) the objective function $f(x)$ is differentiable and quasiconcave in the nonnegative orthant,
(b) each constraint function $g^i(x)$ is differentiable and quasiconvex in the nonnegative orthant,
(c) each point $x^*$ satisfies the Kuhn-Tucker maximum conditions, and
(d) any of the following is satisfied:
  (d.i) $f_j(x^*) < 0$ for at least one variable $x_j$,
  (d.ii) $f_j(x^*) > 0$ for some variable $x_j$ that can take on a positive value without violating the constraints,
  (d.iii) then $n$ derivatives $f_j(x^*)$ are not all zero, and the function $f(x)$ is twice differentiable in the neighborhood of $x^*$, or
  (d.iv) the function $f(x)$ is concave

then $x^*$ gives a global maximum for $f(x)$.

For (c) to be satisfied, the ***constraint qualification*** for a maximum must be satisfied:

(a) $g^i(x)$ is differentiable and quasiconvex for all $i$,
(b) there exists an $x$ in the nonnegative orthant that satisfies all constraints with strict inequalities, and
(c) one of the following is true
  (c.i) $g^i(x)$ is differentiable and convex for all $i$, or
  (c.ii) the partial derivatives of $g^i(x)$ for all $n$ and $i$ are not all zero when evaluated at every point in the feasible set.

Note that if the feasible region is a convex set of linear constraints then the ***constraint qualification*** will be met, which is the case for our classical utility optimization problem.

The Lagrangian for this problem can be written as

$$L = f(x) + \sum_{i=1}^{m} \lambda_i \left(r_i - g^i(x)\right)$$

with the first-order conditions

$\frac{\partial L}{\partial x_l} = \frac{\partial f(x^*)}{\partial x_l} - \sum_{i=1}^{m} \lambda_i{}^* \frac{\partial g^i(x^*)}{\partial x_l} \leq 0$, $\frac{\partial L}{\partial x_l} x_l{}^* = 0$, $x_l{}^* \geq 0$ for $l = 1,\ldots,n$,

$\frac{\partial L}{\partial \lambda_i} = r_i - g^i(x^*) \geq 0$, $\frac{\partial L}{\partial \lambda_i{}^*} \lambda_i{}^* = 0$, $\lambda_i{}^* \geq 0$ for $i = 1,\ldots,m$.

Given the nonlinear program for $x \in \mathbb{R}_+^n$

$$\min_{x \geq 0} f(x) \text{ subject to } g^i(x) \geq r_i \text{ for } i = 1,\dots,\text{m}$$

If the following conditions are satisfied:

(a) the objective function $f(x)$ is differentiable and quasiconvex in the nonnegative orthant,
(b) each constraint function $g^i(x)$ is differentiable and quasiconcave in the nonnegative orthant,
(c) each point $x^*$ satisfies the Kuhn-Tucker maximum conditions, and
(d) any of the following is satisfied:
  (d.i) $f_j(x^*) > 0$ for at least one variable $x_j$,
  (d.ii) $f_j(x^*) < 0$ for some variable $x_j$ that can take on a positive value without violating the constraints,
  (d.iii) then $n$ derivatives $f_j(x^*)$ are not all zero, and the function $f(x)$ is twice differentiable in the neighborhood of $x^*$, or
  (d.iv) the function $f(x)$ is convex

then $x^*$ gives a global minimum for $f(x)$.

For (c) to be satisfied, the ***constraint qualification*** for a minimum must be satisfied:

(a) $g^i(x)$ is differentiable and quasiconcave for all $i$,
(b) there exists an $x$ in the nonnegative orthant that satisfies all constraints with strict inequalities, and
(c) one of the following is true
  (c.i) $g^i(x)$ is differentiable and concave for all $i$, or
  (c.ii) the partial derivatives of $g^i(x)$ for all $n$ and $i$ are not all zero when evaluated at every point in the feasible set.

Note that if the feasible region is a convex set of linear constraints then the ***constraint qualification*** will be met, which is not the case with our classical expenditure minimization problem. That said, since the typical assumptions we impose on utility functions make it quasiconcave and local nonsatiation implies that (c.ii) will hold, the classical expenditure minimization problem does satisfy the ***constraint qualification***.

The Lagrangian for this problem can be written as

$$L = f(x) + \sum_{i=1}^{m} \gamma_i \left( r_i - g^i(x) \right)$$

with the first-order conditions

$\frac{\partial L}{\partial x_l} = \frac{\partial f(x^*)}{\partial x_l} - \sum_{i=1}^{m} \gamma_i{}^* \frac{\partial g^i(x^*)}{\partial x_l} \geq 0, \frac{\partial L}{\partial x_l} x_l{}^* = 0, x_l{}^* \geq 0$ for $l = 1,\dots,n$,

$\frac{\partial L}{\partial \gamma_i} = r_i - g^i(x) \leq 0, \frac{\partial L}{\partial \gamma_i{}^*} \gamma_i{}^* = 0, \gamma_i{}^* \geq 0$ for $i = 1,\dots,m$.