MICROECONOMIC ANALYSIS
ECON 8001-2
Fall 2009

Instructor: Terry Hurley — Due: 10-13-09
TA: Giovann Alarcon

HOMEWORK #5: ANSWERS

Note: When writing up your answers, carefully define all new notation and terms that you introduce, and write in complete sentences and paragraphs.

1. Suppose we have only two consumers labeled ***a*** and ***b***, and two goods labeled 1 and 2. Individual ***a***'s wealth is w^a, while individual ***b***'s wealth is w^b. Individual ***a***'s consumption of good 1 and 2 is denoted by x_1^a and x_2^a, while individual ***b***'s consumption of good 1 and 2 is denoted by x_1^b and x_2^b. Individual ***a***'s preferences are represented by the indirect utility function $v^a(p_1, p_2, w^a) = \frac{w^a}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}$, while individual ***b***'s are represented by the indirect utility function $v^b(p_1, p_2, w^b) = \frac{w^b}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}}$. Both of these indirect utility function are consistent with utility functions for continuous, locally nonsatiated, and strictly convex preference relations.

   (a) Use these indirect utility functions to derive aggregate demand for both goods.
   (b) Is it possible to write this aggregate demand as just a function of aggregate wealth (i.e. $W = w^a + w^b$) and prices? Explain.

ANSWER:

(a)

Roy's Identity implies:

$$x_1^a(p_1, p_2, w^a) = -\frac{\frac{\partial v^a(p_1, p_2, w^a)}{\partial p_1}}{\frac{\partial v^a(p_1, p_2, w^a)}{\partial w^a}} = -\frac{-\frac{1}{3}\frac{w^a}{p_1^{\frac{4}{3}} p_2^{\frac{2}{3}}}}{\frac{1}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}} = \frac{w^a}{3p_1},$$

$$x_2^a(p_1, p_2, w^a) = -\frac{\frac{\partial v^a(p_1, p_2, w^a)}{\partial p_2}}{\frac{\partial v^a(p_1, p_2, w^a)}{\partial w^a}} = -\frac{-\frac{2}{3}\frac{w^a}{p_1^{\frac{1}{3}} p_2^{\frac{5}{3}}}}{\frac{1}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}} = \frac{2w^a}{3p_2},$$

$$x_1^b(p_1, p_2, w^b) = -\frac{\dfrac{\partial v^b(p_1, p_2, w^b)}{\partial p_1}}{\dfrac{\partial v^b(p_1, p_2, w^b)}{\partial w^b}} = -\frac{-\dfrac{2}{3}\dfrac{w^b}{p_1^{\frac{5}{3}} p_2^{\frac{1}{3}}}}{\dfrac{1}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}}} = \frac{2w^b}{3p_1}, \text{ and}$$

$$x_2^b(p_1, p_2, w^b) = -\frac{\dfrac{\partial v^b(p_1, p_2, w^b)}{\partial p_2}}{\dfrac{\partial v^b(p_1, p_2, w^b)}{\partial w^b}} = -\frac{-\dfrac{1}{3}\dfrac{w^b}{p_1^{\frac{2}{3}} p_2^{\frac{4}{3}}}}{\dfrac{1}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}}} = \frac{w^b}{3p_2}.$$

Now that we have individual demands, we can aggregate:

$$x_1(p_1, p_2, w^a, w^b) = x_1^a(p_1, p_2, w^a) + x_1^b(p_1, p_2, w^b) = \frac{w^a}{3p_1} + \frac{2w^b}{3p_1} = \frac{w^a + 2w^b}{3p_1} \text{ and}$$

$$x_2(p_1, p_2, w^a, w^b) = x_2^a(p_1, p_2, w^a) + x_2^b(p_1, p_2, w^b) = \frac{2w^a}{3p_2} + \frac{w^b}{3p_2} = \frac{2w^a + w^b}{3p_2}.$$

(b)

This aggregate demand cannot be written as only a function of prices and wealth because the indirect utility functions are not of the necessary and sufficient Gorman form: $v^i(p_1, p_2, w^i) = a^i(p_1, p_2) + b(p_1, p_2)w^i$. For consumer a, we have $a^a(p_1, p_2) = 0$ and $b(p_1, p_2) = \frac{1}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}$, while for consumer b, we have $a^b(p_1, p_2) = 0$ and $b(p_1, p_2) = \frac{1}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}} \neq \frac{1}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}$.

2. Given the consumers in question 1, consider the social welfare function $U(u^a, u^b) = u^a u^b$.

   (a) Set up the social welfare maximization problem for distributing aggregate wealth W between a and b and derive the first order conditions.
   (b) Use these first order conditions to find the wealth distribution rule that maximizes social welfare (note that the solution will be interior because the objective is concave and the constraint is linear if you set up the problem right). Is this wealth distribution rule homogeneous of degree 1 in prices and wealth?
   (c) Derive the value function for this social welfare optimization problem by substituting the wealth distribution rule back into the objective function.

ANSWER:

(a)

The social welfare maximization problem is

$$\max_{w^a \geq 0,\, w^b \geq 0} U\left(v^a\left(p_1, p_2, w^a\right), v^b\left(p_1, p_2, w^b\right)\right) \text{ subject to } W \geq w^a + w^b.$$

The Lagrangian can be written as

$$L = \left(\frac{w^a}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}\right)\left(\frac{w^b}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}}\right) + g\left(W - w^a - w^b\right),$$

which has the first order conditions

$$\frac{\partial L}{\partial w^a} = \left(\frac{1}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}\right)\left(\frac{w^{b^*}}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}}\right) - g^* \leq 0, \quad \frac{\partial L}{\partial w^a} w^{a^*} = 0, \text{ and } w^{a^*} \geq 0;$$

$$\frac{\partial L}{\partial w^b} = \left(\frac{w^{a^*}}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}\right)\left(\frac{1}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}}\right) - g^* \leq 0, \quad \frac{\partial L}{\partial w^b} w^{b^*} = 0, \text{ and } w^{b^*} \geq 0; \text{ and}$$

$$\frac{\partial L}{\partial g} = W - w^{a^*} - w^{b^*} \geq 0, \qquad \frac{\partial L}{\partial g} g^* = 0, \text{ and } g^* \geq 0.$$

(b)

For an interior solution, $\left(\frac{1}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}\right)\left(\frac{w^{b^*}}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}}\right) = \left(\frac{w^{a^*}}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}\right)\left(\frac{1}{p_1^{\frac{2}{3}} p_2^{\frac{1}{3}}}\right)$ or $w^{b^*} = w^{a^*}$, and $w^{a^*} + w^{b^*} = W$. Solving yields the wealth distribution rule $w^a(p_1, p_2, W) = w^{a^*} = \frac{1}{2}W$ and $w^b(p_1, p_2, W) = w^{b^*} = \frac{1}{2}W$. For this wealth distribution rule to be homogeneous of degree 1 in prices and wealth, $w^a(ap_1, ap_2, aW) = aw^a(p_1, p_2, W)$ and $w^b(ap_1, ap_2, aW) = aw^b(p_1, p_2, W)$ for any $a > 0$, which is clearly the case: $w^a(ap_1, ap_2, aW) = \frac{1}{2}aW = aw^a(p_1, p_2, W)$ and $w^b(ap_1, ap_2, aW) = \frac{1}{2}aW = aw^b(p_1, p_2, W)$

(c)

The value function is $V(p_1, p_2, W) = U\left(v^a\left(p_1, p_2, w^a\left(p_1, p_2, W\right)\right), v^b\left(p_1, p_2, \left(p_1, p_2, W\right)\right)\right) =$
$\left(\frac{W}{2p_1^{\frac{1}{3}}p_2^{\frac{2}{3}}}\right)\left(\frac{W}{2p_1^{\frac{2}{3}}p_2^{\frac{1}{3}}}\right) = \frac{W^2}{4p_1p_2}$.

3. Treating the value function you derived in question 2 (c) as an indirect utility function:
   (a) Under what conditions is using this value function for welfare analysis theoretically justified? Show that this value function satisfies these conditions.
   (b) Use the change in consumer surplus to calculate the welfare effect of a unit tax of $t > 0$ on good 1 such that the price of good after the tax is $p_1' = p_1 + t$. Assume that none of the tax revenues are returned to the consumers.
   (c) Derive the expenditure function for this indirect utility function. How will the change in consumer surplus compare to the equivalent and compensating variations calculated using this expenditure function? You do not need to calculate the equivalent and compensating variations, but you do need to otherwise justify your answer.

ANSWER:
(a)
To be able to theoretically justify using the value function $V(p_1, p_2, W)$ for welfare analysis, we need to have (i) a wealth distribution rule that is homogeneous of degree 1 in p_1, p_2, and W; (ii) $V(p_1, p_2, W)$ satisfy the properties of an indirect utility function (ii.i) homogeneous of degree 0 in in p_1, p_2, and W, (ii.ii) nonincreasing in p_1 and p_2, and strictly increasing in W, (ii.iii) quasiconvex in p_1 and p_2, and (ii.iv) continuous in p_1, p_2, and W; and (iii) $V(p_1, p_2, W)$ to return our aggregate demands when aggregate demand is evaluated at our wealth distribution rule.

We have already argued that (i) is satisfied in question 2 (b).

For (ii.i) to hold, $V(ap_1, ap_2, aW) = V(p_1, p_2, W)$ for $a > 0$: $V(ap_1, ap_2, aW) = \frac{(aW)^2}{4ap_1ap_2} =$
$\frac{W^2}{4p_1p_2} = V(p_1, p_2, W)$.

For (ii.ii), $\frac{\partial V(p_1, p_2, W)}{\partial p_1} = -\frac{W^2}{4p_1^2 p_2} < 0$, $\frac{\partial V(p_1, p_2, W)}{\partial p_2} = -\frac{W^2}{4p_1p_2^2} < 0$, and
$\frac{\partial V(p_1, p_2, W)}{\partial W} = \frac{2W}{4p_1p_2} > 0$, as required.

For (ii.iii), requires $V(p_1, p_2, W)$ to be positive semidefinite in p_1 and p_2 such that for

$$[H_p] = \begin{bmatrix} \frac{\partial^2 V(p_1, p_2, W)}{\partial p_1^{\,2}} & \frac{\partial^2 V(p_1, p_2, W)}{\partial p_1 \partial p_2} \\ \frac{\partial^2 V(p_1, p_2, W)}{\partial p_1 \partial p_2} & \frac{\partial^2 V(p_1, p_2, W)}{\partial p_2^{\,2}} \end{bmatrix}. = \begin{bmatrix} \frac{W^2}{2p_1^{\,3} p_2} & \frac{W^2}{4p_1^{\,2} p_2^{\,2}} \\ \frac{W^2}{4p_1^{\,2} p_2^{\,2}} & \frac{W^2}{2p_1 p_2^{\,3}} \end{bmatrix}$$

$\frac{\partial^2 V(p_1, p_2, W)}{\partial p_1^{\,2}} = \frac{W^2}{2p_1^{\,3} p_2} \geq 0$ and $|H_p| = \left(\frac{W^2}{2p_1^{\,3} p_2} \frac{W^2}{2p_1 p_2^{\,3}} - \left(\frac{W^2}{4p_1^{\,2} p_2^{\,2}} \right)^2 \right) = \frac{3W^4}{16p_1^{\,4} p_2^{\,4}} \geq 0$, which must be true for $p_1 > 0$, $p_2 > 0$, and $W > 0$.

For (ii.iv), inspection should be enough to convince you of continuity.

For (iii), Roy's Identity implies

$$x_1(p_1, p_2, W) = -\frac{\frac{\partial V(p_1, p_2, W)}{\partial p_1}}{\frac{\partial V(p_1, p_2, W)}{\partial w}} = -\frac{-\frac{W^2}{4p_1^{\,2} p_2}}{\frac{2W}{4p_1 p_2}} = \frac{W}{2p_1}, \text{ and}$$

$$x_2(p_1, p_2, W) = -\frac{\frac{\partial V(p_1, p_2, W)}{\partial p_2}}{\frac{\partial V(p_1, p_2, W)}{\partial w}} = -\frac{-\frac{W^2}{4p_1 p_2^{\,2}}}{\frac{2W}{4p_1 p_2}} = \frac{W}{4p_2}.$$

Substituting the wealth distribution rule derived in question 2 (b) into the aggregate demands derived in question 1 (a) yields

$$x_1(p_1, p_2, w^a(p_1, p_2, W), w^b(p_1, p_2, W)) = \frac{\frac{W}{2} + 2\frac{W}{2}}{3p_1} = \frac{W}{2p_1} \text{ and}$$

$$x_2(p_1, p_2, w^a(p_1, p_2, W), w^b(p_1, p_2, W)) = \frac{2\frac{W}{2} + \frac{W}{2}}{3p_2} = \frac{W}{2p_2},$$

which are identical to what we derived using Roy's Identity with the value function from the social welfare optimization problem.

(b)
To calculate the change in consumer surplus from a unit tax of t on good 1, we can subtract the consumer surplus without the tax from consumer surplus with the tax:

$$\Delta\text{CS} = \int_{p_1+t}^{\infty} \frac{W}{2p_1'} dp_1' - \int_{p_1}^{\infty} \frac{W}{2p_1'} dp_1' = \int_{p_1+t}^{\infty} \frac{W}{2p_1'} dp_1' - \int_{p_1+t}^{\infty} \frac{W}{2p_1'} dp_1' - \int_{p_1}^{p_1+t} \frac{W}{2p_1'} dp_1' = -\int_{p_1}^{p_1+t} \frac{W}{2p_1'} dp_1' =$$

$$-\frac{W}{2}\left(\ln\left(p_1+t\right)-\ln\left(p_1\right)\right) = -\frac{W}{2}\ln\left(\frac{p_1+t}{p_1}\right).$$

(c)

To get the expenditure function implied by our value function, we can use duality: $V(p_1, p_2, W^*)$ $= \frac{W^{*2}}{4p_1p_2}$ implies $U^* = \frac{e\left(p_1, p_2, U^*\right)^2}{4p_1p_2}$ or $e\left(p_1, p_2, U^*\right) = 2\sqrt{p_1p_2U^*}$. Note that good 1 is normal. Therefore, $\frac{\partial e\left(p_1', p_2, V\left(p_1, p_2, w^*\right)\right)}{\partial p_1} = h_1(p_1', p_2, V(p_1, p_2, W)) > h_1(p_1', p_2, V(p_1 + t, p_2, W)) =$ $\frac{\partial e\left(p_1', p_2, V\left(p_1+t, p_2, w^*\right)\right)}{\partial p_1}$, which implies $0 > EV > \Delta\text{CS} > CV$ since we have a price increase.