MICROECONOMIC ANALYSIS
ECON 8001-2
Fall 2009

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HOMEWORK #3: ANSWERS

Note: When writing up your answers, carefully define all new notation and terms that you introduce, and write in complete sentences and paragraphs.

1. Let $u(x_1, x_2) = \left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right)^2$ represent a consumer's continuous, locally non-satiated, strictly convex preference relation for $x \in \Re_+^2$. Assume this consumer has income $w > 0$ and faces prices p_1 and p_2 for x_1 and x_2.
   (a) Formulate the consumer's utility maximization problem.
   (b) Derive the Marshallian demand for x_1 and x_2 assuming the solution is interior.
   (c) Derive the individual's indirect utility function.
   (d) Verify that the indirect utility function is (i) homogenous of degree 0 in p_1, p_2, and w, (ii) strictly increasing in w and nonincreasing in p_1 and p_2, and (iii) quasi-convex in p_1 and p_2.

ANSWER:
(a)

The consumer's utility maximization problem can be formulated as

$$\max_{x_1 \geq 0, x_2 \geq 0} \left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right)^2$$
subject to
$$w = p_1 x_1 + p_2 x_2.$$

(b)

The Lagrangian for this utility maximization problem is

$$L = \left(x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}\right)^2 + \lambda(w - p_1 x_1 - p_2 x_2)$$

which has the first-order conditions

(1.1) $\frac{\partial L}{\partial x_1} = \left({x_1^*}^{\frac{1}{2}} + {x_2^*}^{\frac{1}{2}}\right) {x_1^*}^{-\frac{1}{2}} - \lambda^* p_1 \leq 0, \frac{\partial L}{\partial x_1} x_1^* = 0, x_1^* \geq 0$

(1.2) $\frac{\partial L}{\partial x_2} = \left({x_1^*}^{\frac{1}{2}} + {x_2^*}^{\frac{1}{2}}\right) {x_2^*}^{-\frac{1}{2}} - \lambda^* p_2 \leq 0, \frac{\partial L}{\partial x_2} x_2^* = 0, x_2^* \geq 0$

(1.3) $\frac{\partial L}{\partial \lambda} = w - p_1 x_1{}^* - p_2 x_2{}^* = 0$, and $\lambda^* \geq 0$.

If the solution is interior, equation (1.1) and (1.2) imply

(1.4) $\frac{\left(x_1{}^{*\frac{1}{2}} + x_2{}^{*\frac{1}{2}}\right) x_1{}^{*-\frac{1}{2}}}{\left(x_1{}^{*\frac{1}{2}} + x_2{}^{*\frac{1}{2}}\right) x_2{}^{*-\frac{1}{2}}} = \frac{\lambda^* p_1}{\lambda^* p_2}$ or $x_2{}^* = \left(\frac{p_1}{p_2}\right)^2 x_1{}^*$.

Substituting equation (1.4) into equation (1.3) and solving yields

(1.5) $x_1(p_1, p_2, w) = x_1{}^* = \frac{p_2}{p_1} \frac{w}{(p_2 + p_1)}$.

Substituting equation (1.5) into equation (1.4) and solving yields

(1.6) $x_2(p_1, p_2, w) = x_2{}^* = \frac{p_1}{p_2} \frac{w}{(p_2 + p_1)}$.

(c)

The indirect utility function is

(1.7) $$v(p_1, p_2, w) = u(x_1(p_1, p_2, w), x_2(p_1, p_2, w))$$
$$= \left(\left(\frac{p_2}{p_1} \frac{w}{(p_2 + p_1)} \right)^{\frac{1}{2}} + \left(\frac{p_1}{p_2} \frac{w}{(p_2 + p_1)} \right)^{\frac{1}{2}} \right)^2$$
$$= \frac{w}{p_2} + \frac{w}{p_1}.$$

(d)

To see that this indirect utility function is homogeneous of degree 0 in p_1, p_2, and w, note that

$v(ap_1, ap_2, aw) = \frac{\alpha w}{\alpha p_2} + \frac{\alpha w}{\alpha p_1} = \frac{w}{p_2} + \frac{w}{p_1} = v(p_1, p_2, w)$ for any $a > 0$.

To see that the indirect utility function is strictly increasing in w and nonincreasing in p_1 and p_2, note that $\frac{\partial v(p_1, p_2, w)}{\partial w} = \frac{1}{p_2} + \frac{1}{p_1} > 0$, and $\frac{\partial v(p_1, p_2, w)}{\partial p_i} = -\frac{w}{p_i{}^2} < 0$ for all $w > 0$, and $p_i > 0$ where $i = 1$, 2.

To see that the indirect utility function is quasiconvex in p_1 and p_2, note that

$$[H_p] = \begin{bmatrix} \frac{\partial^2 v(p_1, p_2, w)}{\partial p_1{}^2} & \frac{\partial^2 v(p_1, p_2, w)}{\partial p_1 \partial p_2} \\ \frac{\partial^2 v(p_1, p_2, w)}{\partial p_1 \partial p_2} & \frac{\partial^2 v(p_1, p_2, w)}{\partial p_2{}^2} \end{bmatrix}. = \begin{bmatrix} \frac{2w}{p_1{}^3} & 0 \\ 0 & \frac{2w}{p_2{}^3} \end{bmatrix}$$

such that $\frac{\partial^2 v(p_1,p_2,w)}{\partial {p_1}^2} = \frac{2w}{{p_1}^3} > 0$ and $|H_p| = \frac{2w}{{p_1}^3}\frac{2w}{{p_2}^3} > 0$ for all $w > 0$, $p_1 > 0$ and $p_2 > 0$, which implies the Hessian is everywhere positive semidefinite and $v(p_1, p_2, w)$ is convex in p_1 and p_2. Since convex functions are also quasiconvex, $v(p_1, p_2, w)$ is also quasiconvex in p_1 and p_2.

2. Using the utility function in 1.,
   (a) Formulate the consumer's expenditure minimization problem.
   (b) Derive the Hicksian demands for x_1 and x_2 assuming the solution is interior.
   (c) Substitute the indirect utility function from 1.(c) for utility in the Hicksian demands you derived in 2.(b) and compare the results to the Marshallian demands you found in 1.(b). What is the implication of your result?
   (d) Derive the expenditure function.
   (e) Substitute the indirect utility function you derived in 1.(c) for utility in the expenditure function you derived in 2.(d) and simplify as much as you can, which should actually be quite a bit. What does your result imply about the relationship between the expenditure and indirect utility function?

ANSWER:
(a)

The consumer's expenditure minimization problem can be formulated as

$$\min_{h_1 \geq 0, h_2 \geq 0} p_1 h_1 + p_2 h_2$$
$$\text{subject to}$$
$$\left(h_1^{\frac{1}{2}} + h_2^{\frac{1}{2}}\right)^2 \geq u.$$

(b)

The Lagrangian for this problem is

$$L = p_1 h_1 + p_2 h_2 + \gamma\left(u - \left(h_1^{\frac{1}{2}} + h_2^{\frac{1}{2}}\right)^2\right),$$

which has the first-order conditions

(2.1) $\frac{\partial L}{\partial h_1} = p_1 - \gamma^*\left({h_1}^{*\frac{1}{2}} + {h_2}^{*\frac{1}{2}}\right){h_1}^{*-\frac{1}{2}} \geq 0$, $\frac{\partial L}{\partial h_1}{h_1}^* = 0$, $h_1^* \geq 0$,

(2.2) $\frac{\partial L}{\partial h_2} = p_2 - \gamma^*\left({h_1}^{*\frac{1}{2}} + {h_2}^{*\frac{1}{2}}\right){h_2}^{*-\frac{1}{2}} \geq 0$, $\frac{\partial L}{\partial h_2}{h_2}^* = 0$, $h_2^* \geq 0$,

(2.3) $\frac{\partial L}{\partial h_2} = u - \left({h_1}^{*\frac{1}{2}} + {h_2}^{*\frac{1}{2}}\right)^2 \leq 0$, $\frac{\partial L}{\partial \gamma}\gamma^*$, and $g^* \geq 0$.

For an interior solution, equations in (2.1) and (2.2) will hold with equality such that

(2.4) $\frac{p_1}{p_2} = \frac{\gamma^*\left({h_1}^{*\frac{1}{2}}+{h_2}^{*\frac{1}{2}}\right){h_1}^{*-\frac{1}{2}}}{\gamma^*\left({h_1}^{*\frac{1}{2}}+{h_2}^{*\frac{1}{2}}\right){h_2}^{*-\frac{1}{2}}}$ or ${h_2}^* = \left(\frac{p_1}{p_2}\right)^2 {h_1}^*$.

Equation (2.3) will also holds with equality, so substituting equation (2.4) implies

(2.5) $\left({h_1}^{*\frac{1}{2}} + \left(\left(\frac{p_1}{p_2}\right)^2 {h_1}^*\right)^{\frac{1}{2}}\right)^2 = u$ or $h_1(p_1, p_2, u) = {h_1}^* = \frac{u{p_2}^2}{(p_2+p_1)^2}$.

Substituting equation (2.5) into equation (2.4) then implies

(2.6) $h_2(p_1, p_2, u) = {h_2}^* = \frac{u{p_1}^2}{(p_2+p_1)^2}$.

(c)

Substituting the indirect utility function from 1. (c) into the Hicksian demands yields

(2.7) $h_1(p_1, p_2, v(p_1, p_2, w)) = \frac{\left(\frac{w}{p_2}+\frac{w}{p_1}\right){p_2}^2}{(p_2+p_1)^2} = \frac{wp_2}{p_1(p_2+p_1)}$ and

(2.8) $h_2(p_1, p_2, v(p_1, p_2, w)) = \frac{\left(\frac{w}{p_2}+\frac{w}{p_1}\right){p_1}^2}{(p_2+p_1)^2} = \frac{wp_1}{p_2(p_2+p_1)}$,

which are identical to equations (1.5) and (1.6).

The implication of this result is that the Hicksian demand is equal to the Marshallian demand if the level of utility we use to constrain expenditures is equal to the level of utility obtained from the consumer's utility maximization problem.

(d)

The expenditure function is

(2.9)
$$\begin{aligned} e(p_1, p_2, u) &= p_1 h_1(p_1, p_2, u) + p_2 h_2(p_1, p_2, u) \\ &= p_1 \frac{u{p_2}^2}{(p_2+p_1)^2} + p_2 \frac{u{p_1}^2}{(p_2+p_1)^2} \\ &= \frac{up_1p_2}{p_1+p_2}. \end{aligned}$$

(e)

Substituting the indirect utility function from 1. (c) into the expenditure function yields

(2.10) $e(p_1, p_2, v(p_1, p_2, w)) = \frac{\left(\frac{w}{p_2}+\frac{w}{p_1}\right)p_1p_2}{p_1+p_2} = w.$

That is, when we evaluate the expenditure function at the indirect utility function, we get income back from the consumer's utility maximization problem, which indicates that the expenditure function is essentially the inverse of the utility function.

3. Suppose that there are two possible states of the world denoted by A and B. Also, suppose a consumer's continuous, locally non-satiated, and strictly convex preferences for income in these two states of the world can be represented by the utility function $u(w_A, w_B) = a w_A^t + (1 - a)w_B^t$ where $w_A \geq 0$ is income in state A, $w_B \geq 0$ is income in state B, and $1 > a > 0$ and $1 > t > 0$ are parameters. In state A, the consumer is endowed with income of $w_A^e > 0$ dollars. In state B, the consumer is endowed with income of $w_B^e > 0$ dollars. Furthermore, assume that the consumer can buy or sell insurance that pays an indemnity of x dollars in state B (e.g. for $x > 0$, the consumer is purchasing insurance, while for $x < 0$, the consumer is selling insurance). The cost of this insurance is px regardless of whether the state of the world is A or B where $1 > p > 0$ is the price of insurance per dollar of indemnification.

   (a) Formulate the consumer's budget constraint in terms of w_A and w_B remembering that preferences are locally non-satiated and $w_A \geq 0$ and $w_B \geq 0$.
   (b) Formulate the consumer's utility maximization problem given this budget constraint.
   (c) Solve for the consumer's Marshallian demands in terms of w_A and w_B. Are there any conditions under which that consumer will only consume income in state A (e.g. $w_B = 0$)? Are there any conditions under which that consumer will only consume income in state B (e.g. $w_A = 0$)?
   (d) Are these demands homogeneous of degree 0 in p? Explain.
   (e) Assuming an interior solution, under what condition will the optimal demand for income in state A be greater than the optimal demand for income in state B? If a is the probability of state A and $1 - a$ is the probability of state B, what is the economic intuition of your result?

   Hint: There is a hard way to do part (d) and (e), and an easy way. I encourage you to take some time to look for the easy way.

ANSWER:
(a)

Given the consumers purchase or sale of insurance, the consumption of income in state A must satisfy $w_A^e – px \geq w_A \geq 0$, which also implies $x \leq w_A^e / p$. In state B, the consumption of income must satisfy $w_B^e – px + x \geq w_B \geq 0$, which also implies $–x \leq w_B^e / (1 – p)$. Now note that $w_A^e – px \geq w_A$ and $w_B^e – px + x \geq w_B$ can be written as $w_A^e / p \geq w_A / p + x$ and $w_B^e / (1 – p) \geq w_B / (1 – p) – x$. Summing then implies $w_A^e / p + w_B^e / (1 – p) \geq w_A / p + x + w_B / (1 – p) – x$, which simplifies to

(3.1) $(1-p)w_A{}^e + pw_B{}^e \geq (1-p)w_A + pw_B,$

yielding the consumer's budget constraint in terms of w_A and w_B. This budget constraint will hold with equality assuming preferences are locally non-satiated.

Now some questions were raised in class regarding whether or not the budget constraint implied by $w_A^e - px \geq w_A$ and $w_B^e - px + x \geq w_B$ for $-w_B^e / (1-p) \leq x \leq w_A^e / p$ is truly equivalent to equation (3.1). For example, if $p = ½$, $w_A^e = 2$, $w_B^e = 2$, and $x = 1$, $w_A^e - px \geq w_A$ and $w_B^e - px + x \geq w_B$ imply $2 - ½ = 1.5 \geq w_A$ and $2 - ½ + 1 = 2.5 \geq w_B$, while equation (3.1) implies $1 + 1 \geq w_A/2 + w_B/2$ or $4 \geq w_A + w_B$. Now $w_A = 4$ and $w_B = 0$ certainly satisfies $4 \geq w_A + w_B$ but it does not satisfy $1.5 \geq w_A$ and $2.5 \geq w_B$. What is going on here? Are the budget sets equivalent or not? Note that x is a choice variable. Therefore, if we want to consume $w_A = 4$ and $w_B = 0$ but $x = 1$ doesn't work, why not choose $x = -w_B^e / (1-p) = -4$ instead. With $x = -4$, $w_A^e - px \geq w_A$ and $w_B^e - px + x \geq w_B$ imply $2 - (-2) = 4 \geq w_A$ and $2 - (-2) + (-4) = 0 \geq w_B$, which now works.

More formally, define

$$B^0_{p,{w_A}^e,{w_B}^e} = \left\{(w_A, w_B) \in \mathbb{R}^2_+ : w_A \leq {w_A}^e - px \; and \; w_B \leq {w_B}^e + (1-p)x \; for \; all \; x \in \left[-\frac{{w_B}^e}{(1-p)}, \frac{{w_A}^e}{p}\right]\right\}$$

and

$$B^1_{p,{w_A}^e,{w_B}^e} = \{(w_A, w_B) \in \mathbb{R}^2_+ : (1-p)w_A + pw_B \leq (1-p){w_A}^e + p{w_B}^e\}.$$

Proposition: For $({w_A}^e, {w_B}^e) \geq (0,0)$, $({w_A}^e, {w_B}^e) \neq (0,0)$ and $p \in (0,1)$, $(w_A, w_B) \in B^0_{p,{w_A}^e,{w_B}^e}$ if and only if $(w_A, w_B) \in B^1_{p,{w_A}^e,{w_B}^e}$.

Proof:
Suppose that there exists $({w_A}', {w_B}') \in B^0_{p,{w_A}^e,{w_B}^e}$ such that $({w_A}', {w_B}') \notin B^1_{p,{w_A}^e,{w_B}^e}$. By definition, $({w_A}', {w_B}') \in B^0_{p,{w_A}^e,{w_B}^e}$ implies ${w_A}' \leq {w_A}^e - px$ and ${w_B}' \leq {w_B}^e + (1-p)x$ for all $x \in \left[-\frac{{w_B}^e}{(1-p)}, \frac{{w_A}^e}{p}\right]$. $w_A' \leq {w_A}^e - px$ implies $\frac{w_A'}{p} \leq \frac{{w_A}^e}{p} - x$, while ${w_B}' \leq {w_B}^e + (1-p)x$ implies $\frac{{w_B}'}{(1-p)} \leq \frac{{w_B}^e}{(1-p)} + x$. $\frac{w_A'}{p} \leq \frac{{w_A}^e}{p} - x$ and $\frac{{w_B}'}{(1-p)} \leq \frac{{w_B}^e}{(1-p)} + x$ imply $\frac{w_A'}{p} + \frac{{w_B}'}{(1-p)} \leq \frac{{w_A}^e}{p} - x + \frac{{w_B}^e}{(1-p)} + x$ or $(1-p)w_A' + p{w_B}' \leq (1-p){w_A}^e + p{w_B}^e$. But $(1-p)w_A' + p{w_B}' \leq (1-p){w_A}^e + p{w_B}^e$ implies $({w_A}', {w_B}') \in B^1_{p,{w_A}^e,{w_B}^e}$ by definition, which contradicts $({w_A}', {w_B}') \notin B^1_{p,{w_A}^e,{w_B}^e}$.

Now suppose there exist $({w_A}', {w_B}') \in B^1_{p,{w_A}^e,{w_B}^e}$ such that $({w_A}', {w_B}') \notin B^0_{p,{w_A}^e,{w_B}^e}$. By definition, $({w_A}', {w_B}') \in B^1_{p,{w_A}^e,{w_B}^e}$ implies $(1-p){w_A}' + p{w_B}' \leq (1-p){w_A}^e + p{w_B}^e$. By definition, $({w_A}', {w_B}') \notin B^0$ implies ${w_A}' > {w_A}^e - px$ or ${w_B}' > {w_B}^e + (1-p)x$ for all $x \in \left[-\frac{{w_B}^e}{(1-p)}, \frac{{w_A}^e}{p}\right]$.

Suppose ${w_A}' > {w_A}^e - px$ and ${w_B}' > {w_B}^e + (1-p)x$. $w_A' > {w_A}^e - px$ implies $\frac{w_A'}{p} > \frac{{w_A}^e}{p} - x$, while $w_B' > {w_B}^e + (1-p)x$ implies $\frac{{w_B}'}{(1-p)} > \frac{{w_B}^e}{(1-p)} + x$. $\frac{w_A'}{p} > \frac{{w_A}^e}{p} - x$ and $\frac{{w_B}'}{(1-p)} > \frac{{w_B}^e}{(1-p)} + x$ imply $\frac{w_A'}{p} + \frac{{w_B}'}{(1-p)} > \frac{{w_A}^e}{p} - x + \frac{{w_B}^e}{(1-p)} + x$ or $(1-p)w_A' + p{w_B}' > (1-p){w_A}^e + p{w_B}^e$, but this contradicts $({w_A}', {w_B}') \in B^1_{p,{w_A}^e,{w_B}^e}$.

Suppose ${w_A}' > {w_A}^e - px$ and ${w_B}' \leq {w_B}^e + (1-p)x$. $w_A' > {w_A}^e - px$ implies $x > \frac{{w_A}^e - w_A'}{p}$, while $w_B' \leq {w_B}^e + (1-p)x$ implies $\frac{w_B' - {w_B}^e}{(1-p)} \leq x$. $x > \frac{{w_A}^e - w_A'}{p}$ and $\frac{w_B' - {w_B}^e}{(1-p)} \leq x$ imply $\frac{{w_A}^e - w_A'}{p} + \frac{w_B' - {w_B}^e}{(1-p)} < 2x$ or $(1-p){w_A}^e + p{w_B}^e + 2pw_B' < 2p(1-p)x + (1-p)w_A' + pw_B' + 2p{w_B}^e$. $({w_A}', {w_B}') \in B^1_{p,{w_A}^e,{w_B}^e}$ and $(1-p){w_A}^e + p{w_B}^e + 2pw_B' < 2p(1-p)x + (1-p)w_A' + pw_B' + 2p{w_B}^e$ imply $(1-p)w_A' + pw_B' < 2p(1-p)x + (1-p)w_A' + pw_B' + 2p{w_B}^e$ or $-\frac{{w_B}^e}{(1-p)} < x$, which cannot hold for all $x \in \left[-\frac{{w_B}^e}{(1-p)}, \frac{{w_A}^e}{p}\right]$, specifically it does not hold for $x = -\frac{{w_B}^e}{(1-p)}$.

Suppose ${w_A}' \leq {w_A}^e - px$ and $w_B' > {w_B}^e + (1-p)x$. $w_A' \leq {w_A}^e - px$ implies $x \leq \frac{{w_A}^e - w_A'}{p}$, while $w_B' > {w_B}^e + (1-p)x$ implies $\frac{w_B' - {w_B}^e}{(1-p)} > x$. $x \leq \frac{{w_A}^e - w_A'}{p}$ and $\frac{w_B' - {w_B}^e}{(1-p)} > x$ imply $2x < \frac{{w_A}^e - w_A'}{p} + \frac{w_B' - {w_B}^e}{(1-p)}$ or $2p(1-p)x - (1-p){w_A}^e + p{w_B}^e < (1-p)w_A' + pw_B' - 2(1-p)w_A'$. $({w_A}', {w_B}') \in B^1_{p,{w_A}^e,{w_B}^e}$ and $2p(1-p)x - (1-p){w_A}^e + p{w_B}^e < (1-p)w_A' + pw_B' - 2(1-p)w_A'$ imply $2p(1-p)x - (1-p){w_A}^e + p{w_B}^e < (1-p){w_A}^e + p{w_B}^e$ or $x < \frac{{w_A}^e}{p}$, which cannot hold for all $x \in \left[-\frac{{w_B}^e}{(1-p)}, \frac{{w_A}^e}{p}\right]$, specifically it does not hold for $x = -\frac{{w_A}^e}{p}$.

Q.E.D.

(b)

The consumer's utility maximization problem can now be formulated as

$$\max_{w_A \geq 0, w_B \geq 0} \alpha {w_A}^\tau + (1-\alpha){w_B}^\tau$$
subject to
$$(1-p){w_A}^e + p{w_B}^e = (1-p)w_A + pw_B.$$

(c)

The Lagrangian for this problem is

$$L = \alpha {w_A}^\tau + (1-\alpha){w_B}^\tau + \lambda\big((1-p){w_A}^e + p{w_B}^e - (1-p)w_A - pw_B\big)$$

The first order conditions are

(3.2) $\frac{\partial L}{\partial w_A} = \alpha\tau {w_A^*}^{\tau-1} - \lambda^*(1-p) \le 0, \frac{\partial L}{\partial w_A} w_A^* = 0, w_A^* \ge 0,$

(3.3) $\frac{\partial L}{\partial w_B} = (1-\alpha)\tau {w_B^*}^{\tau-1} - \lambda^* p \le 0, \frac{\partial L}{\partial w_B} w_B^* = 0, w_B^* \ge 0,$

(3.4) $\frac{\partial L}{\partial \lambda} = (1-p){w_A}^e + p{w_B}^e - (1-p)w_A^* - pw_B^* = 0$, and $\lambda^* \ge 0$.

For an interior solution, the first equation in (3.2) and (3.3) will hold with equality such that

(3.5) $\frac{\alpha\tau {w_A^*}^{\tau-1}}{(1-\alpha)\tau {w_B^*}^{\tau-1}} = \frac{(1-p)}{p}$ or

(3.6) $w_A^* = w_B^* \left(\frac{(1-p)(1-\alpha)}{p\alpha}\right)^{\frac{1}{\tau-1}}$.

Substituting equation (3.6) into (3.4) and solving then yields

(3.7) $w_B(p, {w_A}^e, {w_B}^e) = w_B^* = \frac{p{w_B}^e + (1-p){w_A}^e}{p + (1-p)^{\frac{\tau}{\tau-1}}\left(\frac{(1-\alpha)}{p\alpha}\right)^{\frac{1}{\tau-1}}}$.

Substituting equation (3.7) back into equation (3.6) yields

(3.8) $w_A(p, {w_A}^e, {w_B}^e) = w_A^* = \frac{p{w_B}^e + (1-p){w_A}^e}{p + (1-p)^{\frac{\tau}{\tau-1}}\left(\frac{(1-\alpha)}{p\alpha}\right)^{\frac{1}{\tau-1}}}\left(\frac{(1-p)(1-\alpha)}{p\alpha}\right)^{\frac{1}{\tau-1}}$.

Alternatively, if $w_A^* > 0$ and $w_B^* = 0$, equation (3.2) will hold with equality, but equation (3.3) may not, which implies $\frac{\alpha\tau {w_A^*}^{\tau-1}}{(1-p)} = \lambda^* \ge \frac{(1-\alpha)\tau {w_B^*}^{\tau-1}}{p}$ or ${w_B^*}^{1-\tau} \ge \frac{(1-p)(1-\alpha)}{\alpha p} {w_A^*}^{1-\tau}$. Since $1 > a > 0$, $1 > t > 0$, and $1 > p > 0$, ${w_B^*}^{1-\tau} \ge \frac{(1-p)(1-\alpha)}{\alpha p} {w_A^*}^{1-\tau}$ implies $0 \ge \frac{(1-p)(1-\alpha)}{\alpha p} {w_A^*}^{1-\tau}$, which contradicts $w_A^* > 0$. Therefore, we cannot have a solution where $w_A^* > 0$ and $w_B^* = 0$. Similar arguments can be made to show that $w_A^* = 0$ and $w_B^* > 0$ cannot be true.

Since preferences are non-satiated, $w_A^e > 0$, and $w_B^e > 0$, Walras Law also allows us to eliminate $w_A^* = 0$ and $w_B^* = 0$. Therefore, any solution must be interior.

(d)

If these Marshallian demands are homogeneous of degree 0 in p then $w_A(\varepsilon p, {w_A}^e, {w_B}^e) = w_A(p, {w_A}^e, {w_B}^e)$ and $w_B(\varepsilon p, {w_A}^e, {w_B}^e) = w_B(p, {w_A}^e, {w_B}^e)$ for any $e > 0$. From equation (3.5), we know $\frac{\alpha\tau w_A(p, {w_A}^e, {w_B}^e)^{\tau-1}}{(1-\alpha)\tau w_B(p, {w_A}^e, {w_B}^e)^{\tau-1}} = \frac{(1-p)}{p}$ and $\frac{\alpha\tau w_A(\varepsilon p, {w_A}^e, {w_B}^e)^{\tau-1}}{(1-\alpha)\tau w_B(\varepsilon p, {w_A}^e, {w_B}^e)^{\tau-1}} = \frac{(1-\varepsilon p)}{\varepsilon p}$. Suppose $\varepsilon > 1$, then $\frac{(1-p)}{p} > \frac{(1-\varepsilon p)}{\varepsilon p}$ such that $\frac{\alpha\tau w_A(p, {w_A}^e, {w_B}^e)^{\tau-1}}{(1-\alpha)\tau w_B(p, {w_A}^e, {w_B}^e)^{\tau-1}} > \frac{\alpha\tau w_A(\varepsilon p, {w_A}^e, {w_B}^e)^{\tau-1}}{(1-\alpha)\tau w_B(\varepsilon p, {w_A}^e, {w_B}^e)^{\tau-1}}$ or $\frac{w_A(p, {w_A}^e, {w_B}^e)}{w_A(\varepsilon p, {w_A}^e, {w_B}^e)} > \frac{w_B(p, {w_A}^e, {w_B}^e)}{w_B(\varepsilon p, {w_A}^e, {w_B}^e)}$ which cannot be true if $w_A(\varepsilon p, {w_A}^e, {w_B}^e) = w_A(p, {w_A}^e, {w_B}^e)$ and $w_B(\varepsilon p, {w_A}^e, {w_B}^e) = w_B(p, {w_A}^e, {w_B}^e)$. While p is the implicit price of w_B, it is not the implicit price of w_A. Therefore, multiplying p by a constant does not increase all prices and wealth proportionally, so the demand will not be homogenous of degree 0 in p.

There is however another way we could formulate the problem in order to ensure Marshallian demands are homogeneous of degree 0 in prices. Let w be income available to the consumer to buy two types of securities. Let $w_A \geq 0$ be the number of securities purchased that yield a dollar when state A occurs and $w_B \geq 0$ be the number of securities purchased that yield a dollar when state B occurs. Let $p_A > 0$ and $p_B > 0$ be the market price of w_A and w_B. Our budget constraint can now be written as $w \geq p_A\, w_A + p_B\, w_B$. Given our endowments and assuming our consumer can buy or sell securities, $w = p_A\, w_A{}^e + p_B\, w_B{}^e$, which allows our optimization problem to be written as

$$\max_{w_A \geq 0, w_B \geq 0} \alpha w_A{}^\tau + (1-\alpha) w_B{}^\tau$$
$$\text{subject to}$$
$$p_A w_A{}^e + p_B w_B{}^e = p_A w_A + p_B w_B.$$

The Lagrangian for this problem is

$$L = \alpha w_A{}^\tau + (1-\alpha) w_B{}^\tau + \lambda(p_A w_A{}^e + p_B w_B{}^e - p_A w_A - p_B w_B),$$

yielding the first-order conditions

(3.9) $\frac{\partial L}{\partial w_A} = \alpha\tau w_A{}^{*\tau-1} - \lambda^* p_A \leq 0, \frac{\partial L}{\partial w_A} w_A{}^* = 0, w_A{}^* \geq 0,$

(3.10) $\frac{\partial L}{\partial w_B} = (1-\alpha)\tau w_B{}^{*\tau-1} - \lambda^* p_B \leq 0, \frac{\partial L}{\partial w_B} w_B{}^* = 0, w_B{}^* \geq 0,$

(3.11) $\frac{\partial L}{\partial \lambda} = p_A w_A{}^e + p_B w_B{}^e - p_A w_A{}^* - p_B w_B{}^* = 0$, and $\lambda^* \geq 0$.

As before, it is possible to show that the solution to this problem must be interior, so equation (3.9) and (3.10) imply

(3.12) $\frac{\alpha\tau w_A{}^{*\tau-1}}{(1-\alpha)\tau w_B{}^{*\tau-1}} = \frac{p_A}{p_B},$

While equation (3.11) implies

(3.11') $p_A w_A{}^e + p_B w_B{}^e = p_A w_A{}^* + p_B w_B{}^*.$

Now if we multiply prices, p_A and p_B, by any $e > 0$, it should be clear that equation (3.12) and (3.11') will not change because e will simply divide out. Therefore, for this formulation of the problem the Marshallian demands will be homogeneous of degree 0 in prices.

Question: What is going on here? Are the problems the same or not?

Actually, the problems are the same with one small exception. In the original formulation of the problem, we essentially restricted $p_A = 1 - p$ and $p_B = p$ or $p_A + p_B = 1$. By imposing this restriction, we essentially set relative prices to something specific: $p_A/p_B = (1 - p)/p$.

(e)

Note that $w_A{}^* > w_B{}^*$ implies $\frac{w_A{}^*}{w_B{}^*} > 1$ or from equation (3.6) $\left(\frac{(1-p)(1-\alpha)}{p\alpha}\right)^{\frac{1}{\tau-1}} > 1$. But $\left(\frac{(1-p)(1-\alpha)}{p\alpha}\right)^{\frac{1}{\tau-1}} > 1$ implies $p > 1 - a$ since $1/(t - 1) < 0$. So the optimal level of income in state A will be greater than the optimal level of income in state B if the price of indemnification in state B is greater than the probability of state B. Intuitively, on average, the consumer earns $1 - a$ dollars for every dollar invested in indemnification, which costs p dollars for certain. Therefore, when $p > 1 - a$ there are incentives to sell rather than buy insurance because you will collect more premiums than you payout in indemnities on average, since you only payout in state B it is optimal to have higher income in state A. Alternatively, when $p < 1 - a$ there are incentives to buy rather than sell insurance because you will collect more indemnities than you payout in premiums on average, since you only get payouts in state B it is optimal to have higher income in state B. Note that this qualitative relationship is independent of the degree of risk aversion, which is captured by the parameter t, but you will learn more about this in ECON 8003.