MICROECONOMIC ANALYSIS
ECON 8001-2
Fall 2009

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EXAM #1: ANSWERS

1. Consider the preference relation $\succsim$ on $\mathbb{R}_+^L$. **[30 Points]**
   (a) If this preference relation describes a rational consumer, what two properties must it satisfy? In addition to listing these properties, make sure you carefully define them.
   (b) Use this preference relation to define the strict preference relation $\succ$ on $\mathbb{R}_+^L$.
   (c) If this preference relation is rational, prove that that for all $x, y, z \in \mathbb{R}_+^L$, if $x \succsim y$ and $y \succ z$, then $x \succ z$.

ANSWER:

(a) A rational preference relation is complete and transitive. If a preference relation is complete, then for all $x, y \in \mathbb{R}_+^L$, $x \succsim y$ or $y \succsim x$. A preference relation is transitive if for all $x, y, z \in \mathbb{R}_+^L$ such that $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

(b) For any $x, y \in \mathbb{R}_+^L$, x is strictly preferred to y ($x \succ y$) if and only if $x \succsim y$ and not $y \succsim x$.

(c) A preference relation is rational if it is complete and transitive where completeness and transitivity are defined in part (a). The proof of (c) will proceed by contradiction. Suppose that there is some $x, y, z \in \mathbb{R}_+^L$ such that $x \succsim y$, $y \succ z$ and not $x \succ z$. By the definition in part (b), $y \succ z$ implies $y \succsim z$ and not $z \succsim y$, and not $x \succ z$ implies $x \succsim z$ and $z \succsim x$ or $z \succsim x$ and not $x \succsim z$. Therefore, to complete the proof we must be able to show that (i) $x \succsim y$, $y \succsim z$, not $z \succsim y$, $x \succsim z$ and $z \succsim x$ cannot all be true; and (ii) $x \succsim y$, $y \succsim z$, not $z \succsim y$, $z \succsim x$ and not $x \succsim z$ cannot all be true. By transitivity, $z \succsim x$ and $x \succsim y$ imply $z \succsim y$, but this contradicts not $z \succsim y$, so (i) cannot all be true. By transitivity, $x \succsim y$ and $y \succsim z$ implies $x \succsim z$, but this contradicts not $x \succsim z$, so (ii) cannot all be true.

2. Suppose a consumer's locally nonsatiated, strictly convex, and continuous preference relation $\succsim$ on $\mathbb{R}_+^2$ can be represented by the utility function $u(x_1, x_2) = \left(x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}}\right)^3$. Let $p_1 > 0$ and $p_2 > 0$ be the prices for commodity x_1 and x_2. **[50 Points]**
   (a) Define the expenditure minimization problem for this consumer, setup the Lagrangian, and derive the first order conditions. For an interior solution, what two conditions must be true? What is the economic interpretation of these conditions?

(b) The solution to this problem is $h_1(p_1, p_2, u) = \frac{p_2^{\frac{3}{2}}}{\left(p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}\right)^3}u$ and $h_2(p_1, p_2, u) = \frac{p_1^{\frac{3}{2}}}{\left(p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}\right)^3}u$ where u is the minimally desired level of utility. Use this solution to derive the expenditure function.

(c) List four properties that this expenditure function must satisfy (***you do not need to show that it satisfies these properties***).

(d) Use duality to derive an indirect utility function that is consistent with this expenditure function.

(e) Consider a change in prices from $(p_1, p_2) = (25, 16)$ to $(p_1, p_2) = (25, 9)$. If you were to calculate the change in consumer surplus due to this change in prices (***you do not actually have to calculate it***), what is the most it could be? Explain.

ANSWER:

(a) The expenditure minimization problem is

$$\max_{h_1 \geq 0, h_2 \geq 0} p_1 h_1 + p_2 h_2 \text{ subject to } u \leq \left(h_1^{\frac{1}{3}} + h_2^{\frac{1}{3}}\right)^3.$$

The Lagrangian for the problem is

$$L = p_1 h_1 + p_2 h_2 + g\left(u - \left(h_1^{\frac{1}{3}} + h_2^{\frac{1}{3}}\right)^3\right),$$

which yields the first order conditions:

$$\frac{\partial L}{\partial h_1} = p_1 - g^*\left(h_1^{*\frac{1}{3}} + h_2^{*\frac{1}{3}}\right)^2 h_1^{*-\frac{2}{3}} \geq 0, \quad \frac{\partial L}{\partial h_1} h_1^* = 0, \text{ and } h_1^* \geq 0;$$

$$\frac{\partial L}{\partial h_2} = p_2 - g^*\left(h_1^{*\frac{1}{3}} + h_2^{*\frac{1}{3}}\right)^2 h_2^{*-\frac{2}{3}} \geq 0, \quad \frac{\partial L}{\partial h_2} h_2^* = 0, \text{ and } h_2^* \geq 0; \text{ and}$$

$$\frac{\partial L}{\partial g} = u - \left(h_1^{*\frac{1}{3}} + h_2^{*\frac{1}{3}}\right)^3 \leq 0, \quad \frac{\partial L}{\partial g} g^* = 0, \text{ and } g^* \geq 0.$$

For an interior solution, the first two equations imply $\frac{p_1}{p_2} = \frac{h_2^{*\frac{2}{3}}}{h_1^{*\frac{2}{3}}}$, which says that the consumer will set their marginal rate of substitution equal to the ratio of prices. The third equation implies $\left(h_1^{*\frac{1}{3}} + h_2^{*\frac{1}{3}}\right)^3 = u$, which says that the consumer will choose a bundle of goods such that their level of utility just equals the minimally desired level of utility (i.e. there will be no excess utility).

(b) The expenditure function is defined as $e(p_1, p_2, u) = p_1h_1(p_1, p_2, u) + p_2h_2(p_1, p_2, u)$. Given the solution above $e(p_1, p_2, u) = p_1 \frac{p_2^{\frac{3}{2}}}{\left(p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}\right)^3} u + p_2 \frac{p_1^{\frac{3}{2}}}{\left(p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}\right)^3} u = \frac{p_1 p_2 u}{\left(p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}\right)^2}$.

(c) The expenditure function must be (i) homogeneous of degree 1 in p_1, and p_2; (ii) strictly increasing in u and nondecreasing in p_1 and p_2; (iii) concave in p_1 and p_2; and (iv) continuous in u, p_1, and p_2.

(d) Duality implies that $e(p_1, p_2, v(p_1, p_2, w)) = w$ where w is the consumer's income and $v(p_1, p_2, w)$ is the indirect utility function that represents the consumer's preferences. Using the expenditure function derived in part (b), $\frac{p_1 p_2 v(p_1, p_2, w)}{\left(p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}\right)^2} = w$ or $v(p_1, p_2, w) =$

$\frac{w}{p_1 p_2}\left(p_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}\right)^2$.

(e) There is a quick way and a long way to do this problem. For both ways, it is important to realize the change in consumer surplus will be bounded by the equivalent and compensating variations. From here, the long way is to calculate both and figure out which is bigger. The short way is to realize that we are looking at a decrease in the price of good 2 and that good 2 is normal because $h_2(p_1, p_2, u)$ is increasing in u. Therefore, we know that the equivalent variation will be greater than the compensating variation, so the most that the change in consumer surplus can be is the amount of the equivalent variation: EV((25,16), (25,9), w) =

$e(25, 16, v(25, 9, w)) - w = \frac{25 \times 16}{\left(25^{\frac{1}{2}} + 16^{\frac{1}{2}}\right)^2} \frac{w}{25 \times 9}\left(25^{\frac{1}{2}} + 9^{\frac{1}{2}}\right)^2 - w = \left(2\left(\frac{8}{9}\right)^3 - 1\right)w$.

Alternatively, CV((25,16), (25,9), w) = w - $e(25, 9, v(25, 16, w))$ =

$w - \frac{25 \times 9}{\left(25^{\frac{1}{2}} + 9^{\frac{1}{2}}\right)^2} \frac{w}{25 \times 16}\left(25^{\frac{1}{2}} + 16^{\frac{1}{2}}\right)^2 = w\left(1 - \frac{9^3}{2 \times 8^3}\right)$, such that $\left(2\left(\frac{8}{9}\right)^3 - 1\right)w >$

$w\left(1-\dfrac{9^3}{2\times 8^3}\right)$ as expected. Therefore, the maximum change in the consumer surplus is $\left(2\left(\dfrac{8}{9}\right)^3-1\right)w$.

3. Suppose that there are $i = 1,\ldots,I$ consumers with continuous, strictly convex, and locally nonsatiated preference relations $\succsim^i$ on $\mathbb{R}_+^2$ that can be represented by the indirect utility function $v^i(p_a, p_b, w^i) = \dfrac{w^i}{p_a^{\frac{a+f^i}{2a}}\, p_b^{\frac{a-f^i}{2a}}}$ where a and f^i are parameters such that $a > f^i > -a$; w^i is consumer wealth for consumer i; and $p_a > 0$ and $p_b > 0$ are the prices for the two commodities which are labeled a and b. **[50 Points]**
   (a) Derive the ith consumer's Marshallian demand for commodity ***a***.
   (b) Derive aggregate demand for commodity ***a*** assuming all consumers face the same prices. What additional parameter restrictions would make it possible to write this aggregate demand as a function of prices and just aggregate wealth $\left(W = \sum_{i=1}^{I} w^i\right)$, instead of prices and the distribution of individual wealth? Explain.
   (c) Consider the social welfare function $U(u^1,\ldots, u^I) = \sum_{i=1}^{I} {u^i}^{\frac{1}{2}}$ where u^i is the ith consumer's utility. Find the social welfare maximizing wealth distribution rule for this social welfare function (***Hint: It is an interior solution***).
   (d) Is the optimal wealth distribution rule you derived in part (c) homogeneous of degree 1 in prices and aggregate wealth? Justify your answer.
   (e) Discuss the conditions under which evaluating a tax on commodity ***a*** using the equivalent or compensating variation derived from the aggregate demand you found in part (b) when it is evaluated at the optimal wealth distribution rule you found in part (c) is theoretically justified.

ANSWER:

(a) To derive the Marshallian demand for commodity ***a*** for the ith consumer, we can use Roy's Identity:

$$x_a^{\,i}\left(p_a, p_b, w^i\right) = -\frac{\dfrac{\partial v^i\left(p_a, p_b, w^i\right)}{\partial p_a}}{\dfrac{\partial v^i\left(p_a, p_b, w^i\right)}{\partial w^i}},$$

which implies

$$x_a^{\,i}\left(p_a, p_b, w^i\right) = -\frac{-\frac{a+f^i}{2a}\frac{w^i}{p_a^{\frac{a+f^i}{2a}+1} p_b^{\frac{a-f^i}{2a}}}}{\frac{1}{p_a^{\frac{a+f^i}{2a}} p_b^{\frac{a-f^i}{2a}}}} = \frac{\left(a+f^i\right)w^i}{2ap_a}.$$

(b) To get aggregate demand, we need to sum individual demands across all consumers:

$$X_a\left(p_a, p_b, w^1, \ldots, w^I\right) = \sum_{i=1}^{I} \frac{\left(a+f^i\right)w^i}{2ap_a} = \frac{aW + \sum_{i=1}^{I} f^i w^i}{2ap_a}.$$

The necessary and sufficient conditions required in order for aggregate demand to be a function of prices and just aggregate wealth is that it must be possible to write the indirect utility function in the Gorman form: $v^i(p_a, p_b, w^i) = a^i(p_a, p_b) + b(p_a, p_b)w^i$. Note $v^i(p_a, p_b, w^i)$ is in the form $v^i(p_a, p_b, w^i) = a^i(p_a, p_b)\, b(p_a, p_b)w^i$ where $a^i(p_a, p_b) = \left(\frac{p_b}{p_a}\right)^{\frac{f^i}{2a}}$ and $b(p_a, p_b) = \frac{1}{\left(p_a p_b\right)^{\frac{1}{2}}}$ if $f^i \neq f^j$ for all $i \neq j$. However, if $f^i = f$ for all i, then $v^i(p_a, p_b, w^i)$ would be of the form $v^i(p_a, p_b, w^i) = a^i(p_a, p_b) + b(p_a, p_b)w^i$ where $a^i(p_a, p_b) = 0$ and $b(p_a, p_b) = \frac{1}{p_a^{\frac{a+f}{2a}} p_b^{\frac{a-f}{2a}}}$, which would satisfy the Gorman necessary and sufficient conditions.

(c) To find the welfare maximizing wealth distribution rule given the social welfare function $U(u^1, \ldots, u^I) = \sum_{i=1}^{I} u^{i\frac{1}{2}}$, we need can solve

$$\max_{w^1 \geq 0, \ldots, w^I \geq 0} \sum_{j=1}^{I} v^j\left(p_a, p_b, w^j\right) \text{ subject to } W \geq \sum_{j=1}^{I} w^j .$$

The Lagrangian for this problem can be written as

$$L = \sum_{j=1}^{I}\left(\frac{w^j}{p_a^{\frac{a+f^j}{2a}} p_b^{\frac{a-f^j}{2a}}}\right)^{\frac{1}{2}} + l\left(W - \sum_{j=1}^{I} w^j\right),$$

which, for an interior solution, has the first order conditions

(3.1) $$\frac{\partial L}{\partial w^i} = \frac{1}{2}\frac{1}{p_a^{\frac{a+f^i}{4a}} p_b^{\frac{a-f^i}{4a}}} w^{i^{*-\frac{1}{2}}} - l^* = 0 \text{ and}$$

(3.2) $$W = \sum_{j=1}^{I} w^{j^*}.$$

Rearranging equation (3.1) yields

(3.3) $$w^{i^*} = \left(\frac{p_b}{p_a}\right)^{\frac{f^i}{2a}} \frac{1}{4 l^{*2} (p_a p_b)^{\frac{1}{2}}}.$$

Summing equation (3.3) over all consumers yields

(3.4) $$\sum_{j=1}^{I} w^{j^*} = \sum_{j=1}^{I} \left(\frac{p_b}{p_a}\right)^{\frac{f^j}{2a}} \frac{1}{4 l^{*2} (p_a p_b)^{\frac{1}{2}}} = \frac{1}{4 l^{*2} (p_a p_b)^{\frac{1}{2}}} \sum_{j=1}^{I} \left(\frac{p_b}{p_a}\right)^{\frac{f^j}{2a}}.$$

Substituting equation (3.2) into equation (3.4) and rearranging some more yields

(3.5) $$l^{*2} = \frac{1}{4W (p_a p_b)^{\frac{1}{2}}} \sum_{j=1}^{I} \left(\frac{p_b}{p_a}\right)^{\frac{f^j}{2a}}.$$

Substituting equation (3.5) into equation (3.3) and simplifying yields the desired result:

(3.6) $$w^i(p_a, p_b, W) = \left(\frac{p_b}{p_a}\right)^{\frac{f^i}{2a}} \frac{1}{4 \frac{1}{4W (p_a p_b)^{\frac{1}{2}}} \sum_{j=1}^{I} \left(\frac{p_b}{p_a}\right)^{\frac{f^j}{2a}} (p_a p_b)^{\frac{1}{2}}} = \frac{\left(\frac{p_b}{p_a}\right)^{\frac{f^i}{2a}}}{\sum_{j=1}^{I} \left(\frac{p_b}{p_a}\right)^{\frac{f^j}{2a}}} W.$$

(d) For the optimal wealth distribution rule to be homogenous of degree 1 in prices and aggregate wealth $w^i(tp_a, tp_b, tW) = tw^i(p_a, p_b, W)$ for all i and any $t > 0$. Equation (3.6) implies

$$w^i(tp_a, tp_b, tW) = \frac{\left(\frac{tp_b}{tp_a}\right)^{\frac{f^i}{2a}}}{\sum_{j=1}^{I}\left(\frac{tp_b}{tp_a}\right)^{\frac{f^j}{2a}}} tW = t\,\frac{\left(\frac{p_b}{p_a}\right)^{\frac{f^i}{2a}}}{\sum_{j=1}^{I}\left(\frac{p_b}{p_a}\right)^{\frac{f^j}{2a}}} W = tw^i(p_a, p_b, W)$$

for all i and any $t > 0$. Therefore, our optimal wealth distribution rule is homogenous of degree 1 in prices and aggregate wealth.

(e) For the aggregate demand we found in part (b) to be theoretically valid for welfare analysis given the wealth distribution rule we found in part (c), (i) the wealth distribution rule must be homogeneous of degree one in p_a, p_b, and W; (ii) the value function $V(p_a, p_b, W)$ = $\sum_{j=1}^{I} v^j\left(p_a, p_b, w^j(p_a, p_b, W)\right)$ should satisfy the properties of an indirect utility function: (ii.i) homogeneous of degree zero in p_a, p_b, and W, (ii.ii) nonincreasing in p_a and p_b, and strictly increasing in W, (ii.iii) quasi-convex in p_a and p_b, and (ii.iv) continuous in p_a, p_b, and W; and the value function $V(p_a, p_b, W)$ should give us aggregate demand evaluated at the optimal wealth distribution rule when Roy's Identity is applied.

Now, I did not ask for you to verify all these conditions, but you could have with a substantial amount of more work:

We showed that condition (i) was satisfied in part (d). For (ii), first note that

$$V(p_a, p_b, W) = \sum_{i=1}^{I}\left(\frac{w^i(p_a, p_b, W)}{p_a^{\frac{a+f^i}{2a}} p_b^{\frac{a-f^i}{2a}}}\right)^{\frac{1}{2}} = \sum_{i=1}^{I}\left(\frac{\left(\frac{p_b}{p_a}\right)^{\frac{f^i}{2a}}}{p_a^{\frac{a+f^i}{2a}} p_b^{\frac{a-f^i}{2a}} \sum_{j=1}^{I}\left(\frac{p_b}{p_a}\right)^{\frac{f^j}{2a}}} W\right)^{\frac{1}{2}} = W^{\frac{1}{2}}\left(\sum_{i=1}^{I}\frac{1}{p_a^{\frac{a+f^i}{2a}} p_b^{\frac{a-f^i}{2a}}}\right)^{\frac{1}{2}}$$

.

Alternatively, life will be much easier if we work with the monotonic transformation $V'(p_a, p_b, W) = V(p_a, p_b, W)^2 = \sum_{i=1}^{I}\frac{W}{p_a^{\frac{a+f^i}{2a}} p_b^{\frac{a-f^i}{2a}}}$.

For (ii.i), note that $V'(tp_a,tp_b,tW) = \sum_{i=1}^{I}\frac{(tW)}{(tp_a)^{\frac{a+f^i}{2a}}(tp_b)^{\frac{a-f^i}{2a}}} = \sum_{i=1}^{I}\frac{tW}{t^{\frac{a+f^i}{2a}+\frac{a-f^i}{2a}}p_a^{\frac{a+f^i}{2a}}p_b^{\frac{a-f^i}{2a}}} =$

$\sum_{i=1}^{I}\frac{tW}{tp_a^{\frac{a+f^i}{2a}}p_b^{\frac{a-f^i}{2a}}} = V'(p_a,p_b,W)$ for any $t > 0$ as required.

For (ii.ii): note that

$$\frac{\partial V'(p_a,p_b,W)}{\partial p_a} = \sum_{i=1}^{I}\frac{-\frac{a+f^i}{2a}W}{p_a^{\frac{a+f^i}{2a}+1}p_b^{\frac{a-f^i}{2a}}} < 0,$$

$$\frac{\partial V'(p_a,p_b,W)}{\partial p_b} = \sum_{i=1}^{I}\frac{-\frac{a-f^i}{2a}W}{p_a^{\frac{a+f^i}{2a}}p_b^{\frac{a-f^i}{2a}+1}} < 0, \text{ and}$$

$$\frac{\partial V'(p_a,p_b,W)}{\partial W} = \sum_{i=1}^{I}\frac{1}{p_a^{\frac{a+f^i}{2a}}p_b^{\frac{a-f^i}{2a}}} > 0$$

as required for all $p_a > 0$, $p_b > 0$, and $W > 0$.

For (ii.iii),

$$\begin{bmatrix}\frac{\partial^2 V'(p_a,p_b,W)}{\partial p_a^{\,2}} & \frac{\partial^2 V'(p_a,p_b,W)}{\partial p_b \partial p_a}\\ \frac{\partial^2 V'(p_a,p_b,W)}{\partial p_a \partial p_b} & \frac{\partial^2 V'(p_a,p_b,W)}{\partial p_b^{\,2}}\end{bmatrix} =$$

$$\begin{bmatrix}\sum_{i=1}^{I}\frac{\left(\frac{a+f^i}{2a}\right)\left(\frac{a+f^i}{2a}+1\right)W}{p_a^{\frac{a+f^i}{2a}+2}p_b^{\frac{a-f^i}{2a}}} & \sum_{i=1}^{I}\frac{\left(\frac{a+f^i}{2a}\right)\left(\frac{a-f^i}{2a}\right)W}{p_a^{\frac{a+f^i}{2a}+1}p_b^{\frac{a-f^i}{2a}+1}}\\ \sum_{i=1}^{I}\frac{\left(\frac{a+f^i}{2a}\right)\left(\frac{a-f^i}{2a}\right)W}{p_a^{\frac{a+f^i}{2a}+1}p_b^{\frac{a-f^i}{2a}+1}} & \sum_{i=1}^{I}\frac{\left(\frac{a-f^i}{2a}\right)\left(\frac{a-f^i}{2a}+1\right)W}{p_a^{\frac{a+f^i}{2a}}p_b^{\frac{a-f^i}{2a}+2}}\end{bmatrix}$$

such that

$$\sum_{i=1}^{I}\frac{\left(\frac{a+f^i}{2a}\right)\left(\frac{a+f^i}{2a}+1\right)W}{p_a^{\frac{a+f^i}{2a}+2}\,p_b^{\frac{a-f^i}{2a}}} > 0 \text{ and}$$

$$\sum_{i=1}^{I}\frac{\left(\frac{a+f^i}{2a}\right)\left(\frac{a+f^i}{2a}+1\right)W}{p_a^{\frac{a+f^i}{2a}+2}\,p_b^{\frac{a-f^i}{2a}}}\sum_{i=1}^{I}\frac{\left(\frac{a-f^i}{2a}\right)\left(\frac{a-f^i}{2a}+1\right)W}{p_a^{\frac{a+f^i}{2a}}\,p_b^{\frac{a-f^i}{2a}+2}}-\left(\sum_{i=1}^{I}\frac{\left(\frac{a+f^i}{2a}\right)\left(\frac{a-f^i}{2a}\right)W}{p_a^{\frac{a+f^i}{2a}+1}\,p_b^{\frac{a-f^i}{2a}+1}}\right)^2 =$$

$$\left(\sum_{i=1}^{I}\frac{2}{4p_a^{\frac{f^i}{2a}}p_b^{-\frac{f^i}{2a}}}+\sum_{i=1}^{I}\frac{f^i}{ap_a^{\frac{f^i}{2a}}p_b^{-\frac{f^i}{2a}}}+\sum_{i=1}^{I}\frac{2f^{i2}}{4a^2p_a^{\frac{f^i}{2a}}p_b^{-\frac{f^i}{2a}}}\right)^2\frac{W^2}{p_a^3p_b^3}$$
$$+\left(\sum_{i=1}^{I}\frac{4}{4p_a^{\frac{f^i}{2a}}p_b^{-\frac{f^i}{2a}}}+\sum_{i=1}^{I}\frac{4f^{i2}}{4a^2p_a^{\frac{f^i}{2a}}p_b^{-\frac{f^i}{2a}}}\right)\left(\sum_{i=1}^{I}\frac{1}{4p_a^{\frac{f^i}{2a}}p_b^{-\frac{f^i}{2a}}}-\sum_{i=1}^{I}\frac{f^{i2}}{4a^2p_a^{\frac{f^i}{2a}}p_b^{-\frac{f^i}{2a}}}\right)\frac{W^2}{p_a^3p_b^3}$$
$$+\left(\sum_{i=1}^{I}\frac{2}{4p_a^{\frac{f^i}{2a}}p_b^{-\frac{f^i}{2a}}}-\sum_{i=1}^{I}\frac{f^i}{ap_a^{\frac{f^i}{2a}}p_b^{-\frac{f^i}{2a}}}+\sum_{i=1}^{I}\frac{2f^{i2}}{4a^2p_a^{\frac{f^i}{2a}}p_b^{-\frac{f^i}{2a}}}\right)^2\frac{W^2}{p_a^3p_b^3} > 0$$

as required for quasiconvexity because for all $p_a > 0$, $p_b > 0$, $W > 0$, and $a > f^i$ for all i.

Finally, for (iv), note that

$$-\frac{\frac{\partial V'(p_a,p_b,W)}{\partial p_a}}{\frac{\partial V'(p_a,p_b,W)}{\partial W}}=-\frac{\sum_{i=1}^{I}\frac{-\frac{a+f^i}{2a}W}{p_a^{\frac{a+f^i}{2a}+1}\,p_b^{\frac{a-f^i}{2a}}}}{\sum_{i=1}^{I}\frac{1}{p_a^{\frac{a+f^i}{2a}}\,p_b^{\frac{a-f^i}{2a}}}} = \frac{W}{2ap_a}\left(a+\frac{\sum_{i=1}^{I}f^i\left(\frac{p_b}{p_a}\right)^{\frac{f^i}{2a}}}{\sum_{i=1}^{I}\left(\frac{p_b}{p_a}\right)^{\frac{f^i}{2a}}}\right)$$

and

$$X_a\left(p_a, p_b, w^1(p_a, p_b, W), \ldots, w^I(p_a, p_b, W)\right) = \frac{aW + \sum_{i=1}^{I} f^i w^i(p_a, p_b, W)}{2ap_a} =$$

$$\frac{aW + \sum_{i=1}^{I} f^i \frac{\left(\frac{p_b}{p_a}\right)^{\frac{f^i}{2a}}}{\sum_{j=1}^{I}\left(\frac{p_b}{p_a}\right)^{\frac{f^j}{2a}}} W}{2ap_a} = \frac{W}{2ap_a}\left(a + \frac{\sum_{i=1}^{I} f^i \left(\frac{p_b}{p_a}\right)^{\frac{f^i}{2a}}}{\sum_{j=1}^{I}\left(\frac{p_b}{p_a}\right)^{\frac{f^j}{2a}}}\right)$$

as required for Roy’s Identity.