

MICROECONOMIC ANALYSIS

ECON 8001-2

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Instructor: Terry Hurley

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TA: David Bael

EXAM #1: ANSWERS

1. Consider the preference relation $\underline{\mathbf{f}}$ on \hat{A}_+^L . [15 Points]
- (a) If this preference relation describes a rational consumer, what two properties must it satisfy? In addition to listing these properties, make sure you carefully define them.
 - (b) Use this preference relation to define the indifference relation.
 - (c) If this preference relation is rational, prove that the indifference relation must be transitive.

ANSWER:

(a) A rational preference relation is complete and transitive. If a preference relation is complete, then for all $x, y \in \hat{A}_+^L$, $x \underline{\mathbf{f}} y$ or $y \underline{\mathbf{f}} x$. A preference relation is transitive if for all $x, y, z \in \hat{A}_+^L$ such that $x \underline{\mathbf{f}} y$ and $y \underline{\mathbf{f}} z$, then $x \underline{\mathbf{f}} z$.

(b) For any $x, y \in \hat{A}_+^L$, x is indifferent to y ($x \sim y$) if and only if $x \underline{\mathbf{f}} y$ and $y \underline{\mathbf{f}} x$.

(c) A preference relation is transitive if for all $x, y, z \in \hat{A}_+^L$ such that $x \underline{\mathbf{f}} y$ and $y \underline{\mathbf{f}} z$, then $x \underline{\mathbf{f}} z$. An indifference relation is transitive if for all $x, y, z \in \hat{A}_+^L$ such that $x \sim y$ and $y \sim z$, then $x \sim z$. Note that $x \sim y$ implies $x \underline{\mathbf{f}} y$ and $y \underline{\mathbf{f}} x$, and $y \sim z$ implies $y \underline{\mathbf{f}} z$ and $z \underline{\mathbf{f}} y$ by the definition of indifference. By transitivity, $x \underline{\mathbf{f}} y$ and $y \underline{\mathbf{f}} z$ implies $x \underline{\mathbf{f}} z$. Also by transitivity, $z \underline{\mathbf{f}} y$ and $y \underline{\mathbf{f}} x$ implies $z \underline{\mathbf{f}} x$. Finally, $x \underline{\mathbf{f}} z$ and $z \underline{\mathbf{f}} x$ implies $x \sim z$ by the definition of indifference.

2. Consider a two commodity world with I consumers. The i th consumer has the locally nonsatiated, strictly convex, and continuous preference relation $\underline{\mathbf{f}}_i$ on \hat{A}_+^2 . Assume this preference relation can be represented by the utility function $u_i(x_{1i}, x_{2i}) = x_{1i}^{a_i} x_{2i}^{1-a_i}$ where $1 > a_i > 0$. Let $p_1 > 0$ and $p_2 > 0$ be the prices for commodity x_{1i} and x_{2i} and $w_i > 0$ be an individual's wealth. [25 Points]

(a) Define the utility maximization problem for the i th consumer, setup the Lagrangian, and derive the first order conditions.

(b) The solution to this problem is $x_{1i}(p_1, p_2, w_i) = \frac{a_i w_i}{p_1}$ and $x_{2i}(p_1, p_2, w_i) = \frac{(1-a_i)w_i}{p_2}$. Use

this solution to derive the indirect utility function.

(c) List four properties that this indirect utility function must satisfy. *You do not need to show that it satisfies these properties.*

(d) Find the aggregate demand for commodity 1.

(e) Can this aggregate demand be written as only a function of prices and aggregate wealth

$\left(W = \sum_{i=1}^I w_i \right)$? Use the indirect utility function you derived in part (b) to justify your answer.

ANSWER:

(a) The utility maximization problem is

$$\max_{x_{1i} \geq 0, x_{2i} \geq 0} x_{1i}^{a_i} x_{2i}^{1-a_i} \text{ subject to } w_i \geq p_1 x_{1i} + p_2 x_{2i}.$$

The Lagrangian for the problem is

$$L = x_{1i}^{a_i} x_{2i}^{1-a_i} + I(w_i - p_1 x_{1i} - p_2 x_{2i}),$$

which yields the first order conditions:

$$\frac{\partial L}{\partial x_{1i}} = a_i x_{1i}^{a_i-1} x_{2i}^{1-a_i} - I^* p_1 \leq 0, \quad \frac{\partial L}{\partial x_{1i}} x_{1i}^* = 0, \text{ and } x_{1i}^* \geq 0;$$

$$\frac{\partial L}{\partial x_{2i}} = (1-a_i) x_{1i}^{a_i} x_{2i}^{-a_i} - I^* p_2 \leq 0, \quad \frac{\partial L}{\partial x_{2i}} x_{2i}^* = 0, \text{ and } x_{2i}^* \geq 0; \text{ and}$$

$$\frac{\partial L}{\partial I} = w_i - p_1 x_{1i}^* - p_2 x_{2i}^* \geq 0, \quad \frac{\partial L}{\partial I} I^* = 0, \text{ and } I^* \geq 0.$$

(b) The indirect utility function is defined as $v_i(p_1, p_2, w_i) = u_i(x_{1i}(p_1, p_2, w_i), x_{2i}(p_1, p_2, w_i))$.

$$\text{Given the solution above } v_i(p_1, p_2, w_i) = \left(\frac{a_i w_i}{p_1} \right)^{a_i} \left(\frac{(1-a_i) w_i}{p_2} \right)^{1-a_i} = \frac{a_i^{a_i} (1-a_i)^{1-a_i} w_i}{p_1^{a_i} p_2^{1-a_i}}.$$

(c) The indirect utility function must be (i) homogeneous of degree 0 in w, p_1 , and p_2 ; (ii) strictly increasing in w_i and non-increasing in p_1 and p_2 ; (iii) quasi-convex in p_1 and p_2 ; and continuous in w, p_1 , and p_2 .

(d) Aggregate demand for commodity 1 is defined as $X_1(p_1, p_2, w_1, \dots, w_I) = \sum_{i=1}^I x_{1i}(p_1, p_2, w_i)$

$$\text{or } X_1(p_1, p_2, w_1, \dots, w_I) = \sum_{i=1}^I \frac{a_i w_i}{p_1}.$$

(e) To be able to write aggregate demand as a function of only prices and aggregate wealth, we must be able to write the indirect utility function in the Gorman form: $v_i(p_1, p_2, w_i) = a_i(p_1, p_2) + b(p_1, p_2)w_i$. For the indirect utility function in part (b), $a_i(p_1, p_2) = 0$ and $b(p_1, p_2) =$

$\frac{a_i^{a_i} (1-a_i)^{1-a_i}}{p_1^{a_i} p_2^{1-a_i}}$, but this means $b(p_1, p_2)$ is not independent of i , so it cannot be of the necessary and sufficient Gorman Form.

3. Consider the function $f(p_1, p_2, u) = \frac{8}{3} \left(\frac{3}{5}\right)^{\frac{5}{8}} p_1^b p_2^{\frac{3}{8}} u$ where p_1 and p_2 are prices and u is some minimally desired level of utility. **[20 Points]**
- Use the homogeneity property for an expenditure function that represents a continuous strictly convex, and locally nonsatiated preference relation \underline{f} on \hat{A}_+^2 to determine the value of b that makes this function a valid expenditure function.
 - Derive the Hicksian demand for commodity 1 using this expenditure function.
 - Use duality to find an indirect utility function that is consistent with this expenditure function.
 - Consider the price vectors $p^0 = (3, 3)$ and $p^1 = (15, 3)$. Find the equivalent and compensating variations for a change from prices p^0 to p^1 ? If you were to calculate the change in consumer surplus due to this change in prices, what is the most it could be? Explain.

ANSWER:

(a) For $f(p_1, p_2, u)$ to be a valid expenditure function representing a continuous, strictly convex, and locally nonsatiated preference relation \underline{f} on \hat{A}_+^2 , it must be homogeneous of degree one in prices such that $f(ap_1, ap_2, u) = af(p_1, p_2, u)$ for any $a > 0$. Note that $f(ap_1, ap_2, u) = \frac{8}{3} \left(\frac{3}{5}\right)^{\frac{5}{8}} (ap_1)^b (ap_2)^{\frac{3}{8}} u = \frac{8}{3} \left(\frac{3}{5}\right)^{\frac{5}{8}} a^{b+\frac{3}{8}} p_1^b p_2^{\frac{3}{8}} u = a^{b+\frac{3}{8}} f(p_1, p_2, u)$, so $b = 5/8$ is what we need.

(b) To get the Hicksian demand for commodity 1 from the expenditure function, we need to differentiate with respect to p_1 :

$$h_1(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_1} = \left(\frac{5}{3}\right)^{\frac{3}{8}} \left(\frac{p_2}{p_1}\right)^{\frac{3}{8}} u.$$

(d) Duality implies that $w = e(p_1, p_2, v(p_1, p_2, w))$ or $w = \frac{8}{3} \left(\frac{3}{5}\right)^{\frac{5}{8}} p_1^{\frac{5}{8}} p_2^{\frac{3}{8}} v(p_1, p_2, w)$. Solving

$$\text{then yields } v(p_1, p_2, w) = \left(\frac{5}{3}\right)^{\frac{5}{8}} \frac{3w}{8p_1^{\frac{5}{8}} p_2^{\frac{3}{8}}}.$$

(e) The equivalent variation is defined as

$$EV(p^0, p^1, w) = e(p_1^0, p_2^0, v(p_1^1, p_2^1, w)) - w \text{ or}$$

$$EV(3, 3, 15, 3, w) = \frac{8}{3} \left(\frac{3}{5}\right)^{\frac{5}{8}} 3^{\frac{5}{8}} 3^{\frac{3}{8}} \left(\frac{5}{3}\right)^{\frac{5}{8}} \frac{3w}{8 \times 15^{\frac{5}{8}} 3^{\frac{3}{8}}} = \left(\frac{1}{5^{\frac{5}{8}}} - 1\right) w.$$

The compensating variation is defined as

$$CV(p^0, p^1, w) = w - e(p_1^1, p_2^1, v(p_1^0, p_2^0, w))$$

or

$$CV(3, 3, 15, 3, w) = w - \frac{8}{3} \left(\frac{3}{5}\right)^{\frac{5}{8}} 15^{\frac{5}{8}} 3^{\frac{3}{8}} \left(\frac{5}{3}\right)^{\frac{5}{8}} \frac{3w}{8 \times 3^{\frac{5}{8}} 3^{\frac{3}{8}}} = \left(1 - 5^{\frac{5}{8}}\right) w.$$

Notice that the price of commodity 1 has increased. Also notice that the Hicksian demand for commodity 1 is increasing in utility, so the Marshallian Demand will be normal. With a normal Marshallian Demand we know that $EV(p^0, p^1, w) > CV(p^0, p^1, w)$. We also know that the change in consumer surplus will always be bounded by the equivalent and compensating variations.

Therefore, $\left(\frac{1}{5^{\frac{5}{8}}} - 1\right) w$ is the most that the change in consumer surplus could be.

4. Let $U(u_1, \dots, u_I) = \sum_{i=1}^I \ln(u_i)$ be the a social welfare function and W be aggregate wealth.

[20 Points]

- Using the preferences defined for consumers in problem 2, find the social welfare maximizing wealth distribution rule. Hint: It is an interior solution.
- Find the value function for $U(u_1, \dots, u_I)$ given this optimal wealth distribution rule.
- Use Roy's identity with this value function to calculate the implied aggregate demand for commodity 1.
- How does this demand compare to the aggregate demand for commodity 1 found in problem 2 (d) when you substitute in the optimal wealth distribution rule for individual wealth? Under what circumstances might it be reasonable to use this aggregate demand for welfare analysis?

ANSWER:

- (a) The social welfare maximizing problem can be written as

$$\max_{w_i \geq 0, \dots, w_I \geq 0} U(v_i(p_1, p_2, w_i), \dots, v_I(p_1, p_2, w_i)) \text{ subject to } W \geq \sum_{i=1}^I w_i.$$

The Lagrangian for this problem is

$$L = \sum_{i=1}^I \ln \left(\frac{a_i^{a_i} (1-a_i)^{1-a_i} w_i}{p_1^{a_i} p_2^{1-a_i}} \right) + I \left(W - \sum_{i=1}^I w_i \right) \text{ or}$$

$$L = \sum_{i=1}^I \left\{ \ln(a_i^{a_i} (1-a_i)^{1-a_i}) - \ln(p_1^{a_i} p_2^{1-a_i}) + \ln(w_i) \right\} + I \left(W - \sum_{i=1}^I w_i \right),$$

which yields the first order conditions:

$$\frac{\partial L}{\partial w_i} = \frac{1}{w_i^*} - I^* \leq 0, \quad \frac{\partial L}{\partial w_i} w_i^* = 0, \quad \text{and } w_i^* \geq 0 \text{ for } i = 1, \dots, I \text{ and}$$

$$\frac{\partial L}{\partial I} = W - \sum_{i=1}^I w_i^* \geq 0, \quad \frac{\partial L}{\partial I} I^* = 0, \quad \text{and } I^* \geq 0.$$

For an interior solution $w_i^* = \frac{1}{I^*}$. Summing both sides over all consumers then yields

$$W = \sum_{i=1}^I w_i^* = \sum_{i=1}^I \frac{1}{I^*} = \frac{I}{I^*}, \text{ which implies } \frac{I}{W} = I^* \text{ and } w_i^*(p_1, p_2, W) = \frac{W}{I}.$$

(b) Substituting the optimal wealth distribution rule back into the value function yields:

$$V(p_1, p_2, W) = \sum_{i=1}^I \left(\ln(a_i^{a_i} (1-a_i)^{1-a_i}) - \ln(p_1^{a_i} p_2^{1-a_i}) + \ln\left(\frac{W}{I}\right) \right)$$

(c) Roy's Identity implies

$$X_1(p_1, p_2, W) = - \frac{\frac{\partial V(p_1, p_2, W)}{\partial p_1}}{\frac{\partial V(p_1, p_2, W)}{\partial W}} = - \frac{- \sum_{i=1}^I \frac{a_i p_1^{a_i-1} p_2^{1-a_i}}{p_1^{a_i} p_2^{1-a_i}}}{\sum_{i=1}^I \frac{1}{W}} = \frac{W \sum_{i=1}^I a_i}{I p_1}.$$

The aggregate demand for commodity 1 in problem 2 (d) was $X_1(p_1, p_2, w_1, \dots, w_I) = \sum_{i=1}^I \frac{a_i w_i}{p_1}$.

Substituting in the optimal wealth distribution rule yields $X_1(p_1, p_2, w_1, \dots, w_I) = \sum_{i=1}^I \frac{a_i W}{p_1} =$

$\frac{W \sum_{i=1}^I a_i}{I p_1}$. If social preferences are accurately characterized by our social welfare function, the wealth distribution rule is homogenous of degree 1 in prices and wealth, and the value function

satisfies the properties of an indirect utility function, then we can comfortably use aggregate demand to construct the expenditure function for a representative consumer and use this expenditure function and the value function to calculate the equivalent or compensating variation for use in welfare analysis.