Two-Stage Games

The dynamic games we have considered to date have been pretty straightforward with two players and perfect information. What we want to do now is consider some more examples of dynamic games that are a bit richer in terms of both the number of players and who knows what when. In particular, we want to look at some applications of dynamic games with lots of players and imperfect information. Gibbons refers to these games as Two-Stage games of imperfect information, but this taxonomy is not particularly standardized.

Application: The Bank Run Game

In the bank run game, there are two investors who deposit D in a bank. These deposits must be left in the bank for two periods before they will mature resulting in a return of $2R$ where $R > D$. If the bank is forced to liquidate the assets before maturity, the bank recovers $2r$ to return to its investors where $D > r > D/2$. The investors have two periods in which they can withdraw their deposit. If both investors withdraw their deposits in period 1, both receive $r/2$. If only 1 withdraws its investment in period 1, that investor receives D, while the other receives $2r – D$. If both investors let their deposits ride, the game moves into period 2. In period 2, if both withdraw their investment they receive R. If only one investor withdraws its investment, it receives $2R – D$ while the other receives D. If neither withdraws their investment, the game ends with both receiving R. This game is diagramed in Figure 1 where W denotes a decision to withdraw and N denotes a decision not to withdraw.

This dynamic game has a single subgame (besides the game as a whole) that is only reached if both investors choose not to withdraw their deposits in the first period. Therefore, we can appeal to the subgame perfect equilibrium to eliminate Nash equilibria supported by incredible threats. The normal form for this subgame is summarized as:

		Player 2	
		W	N
Player 1	W	R, R	$2R - D$, D
	N	D, $2R - D$	R, R

Since $R > D$, $2R - D > R$ results in the strictly dominant and unique pure strategy Nash equilibrium (W, W). With this solution, we can prune our tree in Figure 1 to look like the tree in Figure 2. This new game can now be represented in the normal form:

Player 2

		W	N
Player 1	W	r r	$2r - D$ D
	N	D $2r - D$	R R

Since $r < D$ and $2r - D < r$ by assumption, this reduced form game has two pure strategy Nash equilibria (there is also a mixed strategy equilibrium): (W, W) and (N, N).

So what exactly are the equilibrium strategies for each of these equilibria?

Remember a strategy must specify what will be done in each contingency, so a player's strategy set is $\{W, (N, N), (N, W)\}$. If a player chooses withdraw, it knows the game will end immediately, so there are no contingencies for it to plan for, which produces the strategy W. If a player chooses not to withdraw, the game may end immediately if its opponent chooses to withdraw or the game may continue if its opponent chooses not to withdraw. Therefore, when a player chooses not to withdraw, it must plan for the contingency that its opponent chooses not to withdraw. If its opponent chooses not to withdraw, it can choose to either withdraw or not in the second period game, which leads to two different strategies (N, N) and (N, W). For the first of these two strategies, it chooses not to withdraw in the first period and not to withdraw in the second period assuming its opponent chooses not to withdraw in the first period. For the second of these two strategies, it chooses not to withdraw in the first period and to withdraw in the second period assuming its opponent chooses not to withdraw in the first period. Therefore, for the first equilibrium, the equilibrium strategies are (W, W). For the second, they are $((N, W), (N, W))$.

Application: Tariffs and Imperfect International Competition

Lets work through another example in Gibbons where the players change in the second stage of the game. In this game, we have two countries denoted by $i = 1, 2$. Each country has a government, a firm, and consumers for a firm's output. The firms in both countries produce the same output at constant marginal cost c, so there are opportunities for export to the other country. The ith firm's output is the sum of that which it produces for domestic consumption, h_i, and that which it exports, e_j. To protect domestic markets, the government can impose a tariff denoted by t_i on exports from country j. Consumer demand in each country is $P_i(Q_i) = a - Q_i$ where $Q_i = h_i + e_j$. So consumers are essentially treated as passive in this game.

Question: How high of a tariff should each government impose?

The answer to this question is going to depend on the behavior of firms, the objective of each government's policy, and who knows what when.

Lets assume that demand is taken as given and that firms act to maximize profit:

$p_i(t_i, t_j, h_i, e_i, h_j, e_j) = (a - c - h_i - e_j)h_i + (a - c - t_j - h_j - e_i)e_i.$

Governments act to maximize the sum of consumer surplus, firm profits, and tariff revenues:

$W_i(t_i, t_j, h_i, e_i, h_j, e_j) = Q_i^2/2 + p_i(t_i, t_j, h_i, e_i, h_j, e_j) + t_i e_j.$

We will also assume governments set tariffs before production decisions are made. That each government does not know the others choice before setting its tariff, and that each firm does not know the other's output before choosing its own. Therefore, the game goes something like this: 1) Governments choose tariffs simultaneously, 2) tariffs are revealed to the firms, 3) firms choose output simultaneously, and 4) payoffs are realized for all players.

This game has one subgame (besides the game as a whole) that begins with firms' output decisions, so the subgame perfect equilibrium will allow us to narrow the set of Nash equilibrium by getting rid of incredible threats. To find the subgame perfect equilibrium, we must first find a Nash equilibrium for the subgame. That is, we must find each firm's optimal output given the tariffs imposed by each government and the output decision of the other firm. We are all friends here, so we will assume the chosen parameter values result in a nice interior solution to the firm's problem.

From the profit function above, the FOCs for both firms are

$a - c - 2h_i - e_j = 0,$
$a - c - 2h_j - e_i = 0,$
$a - c - t_j - h_j - 2e_i = 0,$ and
$a - c - t_i - h_i - 2e_j = 0.$

The solution to this problem for the *i*th firm is then

$$h_i^* = \frac{a - c + t_i}{3} \text{ and } e_i^* = \frac{a - c - 2t_j}{3}.$$

Substituting back into the profit function results in:

$$p_i(t_i, t_j) = \frac{(a - c + t_i)^2}{9} + \frac{(a - c - 2t_j)^2}{9},$$

which is the solution to the Cournot duopoly model in each market when firms have different marginal costs due to the tariffs imposed by governments.

Consumer surplus is:

$$CS_i(t_i, t_j) = \frac{Q_i^2}{2} = \frac{(2(a - c) - t_i)^2}{18}.$$

Tariff revenue is:

$$TR_i(t_i,t_j) = t_i e_j = \frac{t_i(a-c-2t_i)}{3}.$$

Substituting these expressions back into the government's welfare function and taking the derivative with respect to the tariff yields the FOC:

$$-\frac{2(a-c)-t_i^*}{9}+\frac{2(a-c)+2t_i^*}{9}+\frac{3(a-c)-12t_i^*}{9}=0$$

which does not depend on t_j, the other government's tariff. Therefore, the solution $t_i^* = \frac{a-c}{3}$ for i = 1, 2 is a strictly dominant strategy Nash equilibrium given each firm's Nash behavior in the second stage of the game.

Substituting back into each firm's Nash best response function yields

$$h_i^* = \frac{4(a-c)}{9} \text{ and } e_i^* = \frac{a-c}{9}.$$

The firms produce more for the domestic market than the export market due to the tariff, a result that makes good intuitive sense.

Gibbons goes on to argue that imposing no tariff would be better from a social point of view and that offering export subsidies would be even better. Can you think of a good reason for why this would be the case?

However, another interesting question to ask is what would happen if the governments only cared about themselves and acted to maximize tariff revenues? Its FOC would then be

$$\frac{(a-c)-4t_i^*}{3}=0,$$

resulting in the optimal policy $t_i^* = (a - c)/4$. The firm's optimal quantities would be $h_i^* = \frac{5(a-c)}{12}$ and $e_i^* = \frac{a-c}{6}$. This policy encourages more exports and fewer domestic sales by setting a lower tariff. What does this say about the export demand elasticity?

What if the government ignored consumer concerns? Its FOC would then be

$$\frac{2(a-c)+2t_i^*}{9}+\frac{3(a-c)-12t_i^*}{9}=0,$$

resulting in the optimal policy $t_i^* = (a - c)/2$. The firm's optimal quantities would be $h_i^* = \frac{(a-c)}{2}$ and $e_i^* = 0$. This policy results in a high enough tariff to eliminate all trade and provides the domestic firm with monopoly power.

What if the government ignored producer concerns? Its FOC would then be

$$-\frac{2(a-c)-t_i^*}{9}+\frac{3(a-c)-12t_i^*}{9}=0,$$

resulting in the optimal policy being $t_i^* = (a - c)/11$. The firm's optimal quantities would be $h_i^* = \frac{12(a-c)}{33}$ and $e_i^* = \frac{9(a-c)}{33}$. This policy results in a lower tariff and firms supplying less to the domestic market and more to the export market.

Application: Tournaments (Lazear & Rosen, 1981)

Consider two workers and their boss. The boss wants to get the most out of his workers, while paying them as little as possible. Assume the *i*th worker's output is $y_i = e_i + e_i$ where e_i is effort and e_i is a mean zero random noise. The basic idea is that the boss does not know an individual's effort precisely from observing that individual's output. Workers choose effort simultaneously and e_1 and e_2 are independent draws from the distribution $f(e)$. The boss decides to motivate these workers by offering to pay the most productive worker a wage of w_H. The least productive worker gets a wage of w_L where $w_H > w_L$. The bosses payoff is output minus the wages paid to the worker, $y_1 + y_2 – (w_H + w_L)$. On average, he earns $e_1 + e_2 – (w_H + w_L)$. A workers payoff is equal to his wage minus the cost of his effort, $g(e_i)$ where $g'(e_i) > 0$ and $g''(e_i) > 0$.

Question: How much should the boss pay the most and least productive worker? How much effort should each worker expend?

To summarize, the boss chooses the wage scale for the most and least productive worker and reveals this wage scale to the workers. The workers then simultaneously determine how much effort to devote to work. Each worker then experiences a productivity shock. The boss then observes output and pays each worker according to the pay scale.

This game can again be solved using subgame perfection. The place to start is where each worker chooses how much effort to devote to work given the boss's pay schedule. Worker *i*'s expected payoff at this stage will be

$w_H\Pr(y_i(e_i) > y_j(e_j)) + w_L (1 - \Pr(y_i(e_i) > y_j(e_j))) - g(e_i) =$
$(w_H - w_L)\Pr(y_i(e_i) > y_j(e_j)) + w_L - g(e_i)$.

The resulting FOC is

$$(w_H - w_L)\frac{\partial \Pr\left(y_i(e_i) > y_j(e_j)\right)}{\partial e_i} = g'(e_i).$$

The worker sets the expected increase in its wage equal to the marginal disutility of working harder. Note that $\Pr(y_i(e_i) > y_j(e_j)) = \Pr(e_i + e_i > e_j + e_j) = \Pr(e_i > e_j - e_i + e_j)$. Bayes rule then implies

$\Pr(e_i > e_j - e_i + e_j) = \Pr(e_i > e_j - e_i + e_j | e_j) \Pr(e_j) = \int \left(1 - F\left(e_j - e_i + e_j\right)\right) f\left(e_j\right) de_j$

where $F(\cdot)$ is the CDF of $f(e)$. The FOC can now be written as

$$(w_H - w_L)\int f\left(e_j - e_i + e_j\right) f\left(e_j\right) de_j = g'(e_i).$$

Solving the FOC for both players is a difficult task given the ugliness of these conditions. It is hard to say whether or not a unique solution will exist given the nonlinearity of the FOCs in e_i and e_j. So what is an applied economist to do? Since both players are identical, why not assume they will behave identically, $e_i = e_j = e^*$? If this is the case, the first order condition becomes

$$(w_H - w_L)\int f\left(e_j\right)^2 de_j = g'(e^*).$$

With this expression it is straightforward to show that effort is increasing in the difference in wages $\left(\frac{de^*}{d(w_H - w_L)} = \frac{\int f\left(e_j\right)^2 de_j}{g''(e^*)} > 0\right)$ and decreasing in the variance of the productivity shock when that shock is normally distributed (note that $\int f\left(e_j\right)^2 de_j = \int \left(\frac{1}{\sqrt{2p}\, s} e^{-\frac{e_j^2}{2s^2}}\right)^2 de_j =$

$$\int \frac{1}{2p s^2} e^{-\frac{e_j^2}{s^2}} e\, de_j = \frac{\sqrt{2p}\,\frac{s}{\sqrt{2}}}{2p s^2}\int \frac{1}{\sqrt{2p}\,\frac{s}{\sqrt{2}}} e^{-\frac{e_j^2}{2\left(\frac{s}{\sqrt{2}}\right)^2}} de_j = \frac{1}{2\sqrt{p}\, s}).$$

What else is nice about the symmetric equilibrium is that it allows us to say that each individual has the same chance of winning the tournament. If U_a is the value to each worker of alternative employment, this result implies the boss must choose wages such that

$(w_H + w_L)/2 - g(e^*) \geq U_a$

or else its workers will go elsewhere. The boss's problem is then to maximize

$2e^* - (w_H + w_L)$ subject to $(w_H + w_L)/2 - g(e^*) \geq U_a$. Let $\Delta w = w_H - w_L$. The lagrangian can then be written as

$$L = 2e^*(\Delta w) - (w_H + w_L) + l\left(\frac{w_H + w_L}{2} - g(e^*(\Delta w)) - U_a\right)$$
$$= 2e^*(\Delta w) - (\Delta w + 2w_L) + l\left(\frac{\Delta w + 2w_L}{2} - g(e^*(\Delta w)) - U_a\right),$$

which yields the first order conditions with respect to Δw and w_L:

$$\frac{\partial L}{\partial \Delta w} = 2e^{*\prime}(\Delta w) - 1 + \frac{l}{2} - l g'(e^*(\Delta w)) e^{*\prime}(\Delta w) \leq 0, \quad \Delta w \frac{\partial L}{\partial \Delta w} = 0, \quad \Delta w \geq 0,$$
$$\frac{\partial L}{\partial w_L} = -2 + l \leq 0, \quad w_L \frac{\partial L}{\partial w_L} = 0, \quad w_L \geq 0,$$
$$\frac{\partial L}{\partial l} = \frac{\Delta w + 2w_L}{2} - g(e^*(\Delta w)) - U_a \geq 0, \quad l\frac{\partial L}{\partial l} = 0, \quad \text{and } l \geq 0.$$

Assuming we have an interior solution (i.e. $\Delta w > 0$ and $w_L > 0$), these equation imply $\lambda = 2$, $g'(e^*) = 1$, and $(w_H + w_L)/2 - g(e^*) = U_a$. Since $g'(e^*) = 1$ and $(w_H - w_L)\int f(e_j)^2 de_j = g'(e^*)$,

$\Delta w = w_H - w_L = \dfrac{1}{\int f(e_j)^2 de_j}$ or $w_H = w_L + \dfrac{1}{\int f(e_j)^2 de_j}$. Substituting into

$\dfrac{\Delta w + 2w_L}{2} - g(e^*(\Delta w)) = U_a$, then yields $w_L = U_a + g(e^*) - \dfrac{1}{2\int f(e_j)^2 de_j}$ and

$w_H = U_a + g(e^*) + \dfrac{1}{2\int f(e_j)^2 de_j}$.

For example, suppose $g(e) = \mathrm{e}^{ae}$, such that $g'(e) = a\mathrm{e}^{ae} = 1$ or $e = -\ln(a)/a$. The resulting equilibrium wage scale is then $w_L = U_a + \dfrac{1}{a} - \dfrac{1}{2\int f(e)^2 de}$ and $w_H = U_a + \dfrac{1}{a} + \dfrac{1}{2\int f(e)^2 de}$. Or,

if the shock is normally distributed with mean 0 and variance s^2, $w_L = U_a + \dfrac{1}{a} - \sqrt{p}\, s$

and $w_H = U_a + \dfrac{1}{a} + \sqrt{p}\, s$.

Application: Rent Seeking

Recall that the purpose of rent seeking games is to understand how much individuals will waste trying to secure the right to a valuable resource. To answer this question, we found the Cournot and Stackelberg equilibria for the standard Tullock rent seeking game with asymmetric payoffs. In the Cournot simultaneous move equilibrium, rent dissipation was $x_1 * + x_2 * = \frac{V_1 V_2}{V_1 + V_2}$ where x_1* and x_2* were equilibrium effort for player 1 and 2 and V_1 and V_2 were the value of securing the rights to the resource to players 1 and 2. We then asked how this result would change if we let player 1 moved first, which resulted in the Stackelberg type of equilibrium where rent dissipation was equal to $x_1 * + x_2 * = \frac{V_1}{2}$. Comparing rent dissipation in the Cournot equilibrium to rent dissipation in the Stackelberg equilibrium indicated that the Cournot equilibrium dissipates more (less) rent if $V_2 > (<) V_1$. Therefore, one might conclude that we want a policy that does not allow the player with the highest value to go first and may indeed allow the player with the lowest value to move first.

Question: Is there really a need for such policy or can we believe individuals will play this game in an efficient manner without regulatory mandates?

Suppose that we frame the rent seeking game in the context of a two-stage game where in the first stage players choose who will move first and in the second stage players decide how much effort to invest in the order determined in the first stage of the game. To be specific, both players simultaneously pick who they want to move first in the rent seeking game. If player 1 and 2 choose player 1, player 1 gets to move first in the second stage of the game. If they both choose 2, then 2 gets to move first in the second stage. If one player chooses 1 and the other chooses 2, then they must move simultaneously in the second stage.

Subgame perfection is again the relevant solution concept. Therefore, we begin by solving three different subgames: (i) the one where players move simultaneously, (ii) the one where player 1 moves first, and (iii) the one where player 2 moves first. We have done this already, so lets just write down what we previously found.

For subgame (i), $x_i * = V_i \frac{V_i V_j}{(V_i + V_j)^2}$ and $U_i * = V_i \left\{ \frac{V_i^2}{(V_i + V_j)^2} \right\}$.

For subgame (ii), $x_1 * = V_1 \frac{V_1}{4V_2}$, $x_2 * = \frac{V_1}{2} - \frac{V_1^2}{4V_2}$, $U_1 * = V_1 \frac{V_1}{4V_2}$, and $U_2 * = V_2 - V_1 + \frac{V_1^2}{4V_2}$.

For subgame (iii), $x_1 * = \frac{V_2}{2} - \frac{V_2^2}{4V_1}$, $x_2 * = V_2 \frac{V_2}{4V_1}$, $U_1 * = V_1 - V_2 + \frac{V_2^2}{4V_1}$, and $U_2 * = V_2 \frac{V_2}{4V_1}$.

We can now plug these payoffs into the first stage of the game in the table below:

		Player 2: Vote for 1	Player 2: Vote for 2
Player 1	Vote for 1	$V_1 \frac{V_1}{4V_2}$; $V_2 - V_1 + \frac{V_1^2}{4V_2}$	$V_1 \left\{ \frac{V_1^2}{(V_1+V_2)^2} \right\}$; $V_2 \left\{ \frac{V_2^2}{(V_1+V_2)^2} \right\}$
	Vote for 2	$V_1 \left\{ \frac{V_1^2}{(V_1+V_2)^2} \right\}$; $V_2 \left\{ \frac{V_2^2}{(V_1+V_2)^2} \right\}$	$V_1 - V_2 + \frac{V_2^2}{4V_1}$; $V_2 \frac{V_2}{4V_1}$

Note that $V_i \frac{V_i}{4V_j} \geq V_i \left\{ \frac{V_i^2}{(V_i+V_j)^2} \right\}$, a player is never worse off leading as oppose to playing the simultaneous move game. Therefore, we can rule out (2, 1) and (1, 2) from our set of possible pure strategy Nash equilibria. We can also show player ***i*** is better off letting player ***j*** move first when $V_i > V_j$ (See sketch of proof below). If $V_j > V_i$, player ***i*** is better off moving simultaneously rather than letting player *j* move first. Therefore, if $V_1 > V_2$, neither player 1 or 2 will want to deviate from (2, 2), but player 2 will want to deviate from (1,1). Therefore, there is a pure strategy Nash equilibrium (2, 2). Alternatively, if $V_2 > V_1$, neither player 1 or 2 will want to deviate from (1,1), but player 1 will want to deviate from (2, 2). Therefore, there is a pure strategy Nash equilibrium (1,1). In either of these subgame perfect Nash equilibria, the player who values the prize least moves first, which results in the least amount of rent dissipation. So, if players are left to the own devices, they will likely choose to play the rent seeking contest in a manner that dissipates the least amount of rent.

Proposition: If $V_i > V_j$ for an interior solution, then player ***i*** is better off letting player ***j*** be a Stackelberg leader rather than playing a simultaneous move Cournot game.

Let $r_i = V_i / V_j$, such that $U_i^C = V_i \left\{ \frac{V_i^2}{(V_i+V_j)^2} \right\} = V_i \left\{ \frac{r_i^2}{(r_i+1)^2} \right\}$ and

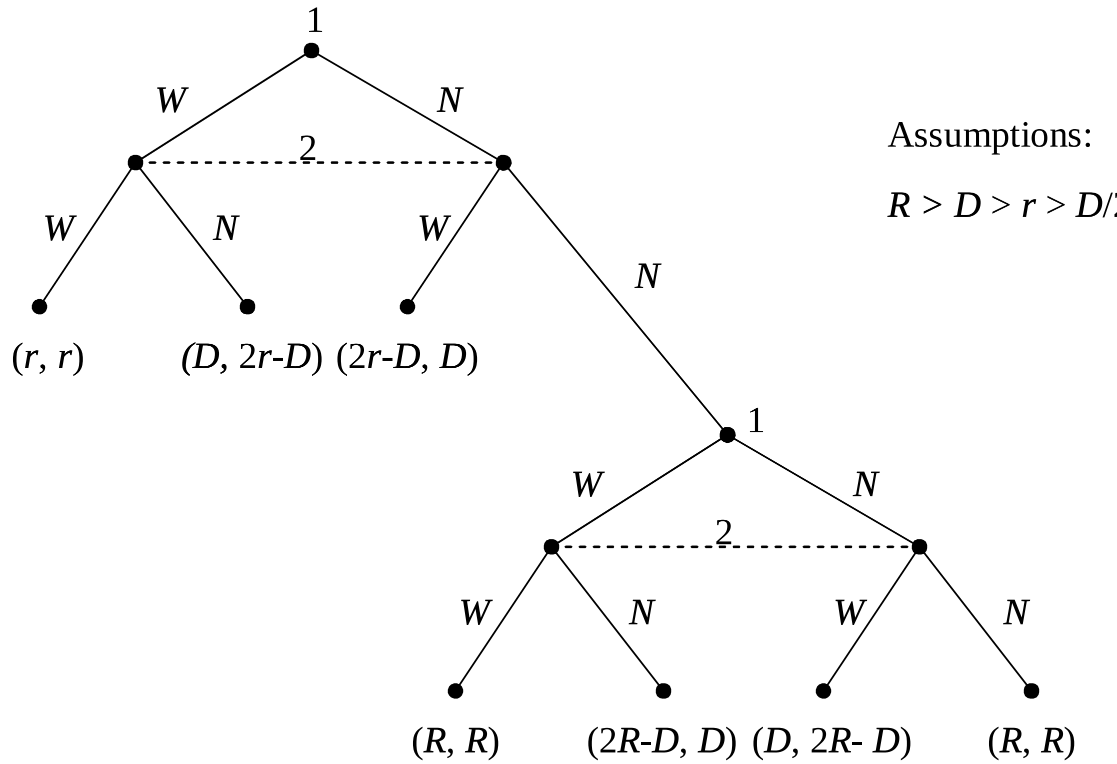
$U_i^{SF} = V_i \left\{ 1 - r_i^{-1} + \frac{1}{4} r_i^{-2} \right\} = V_i \left\{ 1 - \frac{1}{2} r_i^{-1} \right\}^2$. U_i^C >(<) U_i^{SF} when $V_i \left\{ \frac{r_i^2}{(r_i+1)^2} \right\}$ >(<)

$V_i \left\{ 1 - \frac{1}{2} r_i^{-1} \right\}^2 \Rightarrow r_i >(<) (r_i+1)\left(1 - \frac{1}{2} r_i^{-1}\right) \Rightarrow r_i^2 >(<) (r_i+1)\left(r_i - \frac{1}{2}\right) \Rightarrow r_i^2 >(<)$

$r_i^2 + \frac{1}{2} r_i - \frac{1}{2} \Rightarrow 1 >(<) r_i = V_i / V_j \Rightarrow V_j >(<) V_i$. Therefore, when $V_j > V_i$, $U_i^C > U_i^{SF}$ and player

i prefers the Cournot game to following in the Stackelberg game. Alternatively, when $V_i > V_j$, $U_i^{SF} > U_i^C$ and player i prefers following in the Stackelberg game to the Cournot game.

Figure 1: Bank Run Game



Assumptions:

$R > D > r > D/2$

Figure 2: Bank Run Game After Solving For the Second Period Subgame