**Subgame Perfect Equilibrium**

**Objective: Understand the rationale for the subgame perfect refinement to the Nash equilibrium.**

Before going into too much detail about what a subgame perfect equilibrium is, it is useful to think about why the Nash equilibrium is not always enough. Consider the entry deterrence game in Figure 1. There are two players. The first player is a firm that is thinking about entering a market currently dominated by the second player who can be thought of as a monopolist. If Player 1 enters the market, Player 2 can respond with a low noncompetitive duopoly output or could compete more aggressively with higher output to punish Player 1 for its decision to enter the market. The stylized payoffs in Figure 1 indicate that Player 1 prefers to enter the market only if the monopolist responds with a noncompetitive output. Player 2 prefers that Player 1 does not enter the market, but if it does Player 2 is better off responding with a noncompetitive output. The table below summarizes the extensive form game in Figure 1 in normal form.

| | | Player 2: If Enter, Low Output ($s_L$) | Player 2: If Enter, High Output (1-$s_L$) |
|---|---|---|---|
| Player 1 | Enter ($s_E$) | 90, 50 | 60, 10 |
| | Do Not Enter(1-$s_E$) | 70, 60 | 70, 60 |

What are the Nash Equilibria for this game?

{($s_E$ = 1, 1-$s_E$ = 0), ($s_L$ = 1, 1-$s_L$ = 0)} and {($s_E$ = 0, 1-$s_E$ = 1), ($s_L$ < 1/3, 1-$s_L$)}.

That is, there are two pure strategy equilibria and a host of mixed strategy equilibria. The Nash equilibrium does not get us very far in terms of a unique prediction.

Is there any reason to believe one of these Nash equilibria is more likely to be played than the others?

Suppose Player 1 chose to Enter. If this is the case, Player 2 does best by choosing a Low Output. If Player 2 is better off choosing a Low Output when Player 1 chooses Enter, then Player 1's choice is really between 90 from choosing Enter and 70 from choosing not to enter. So, why should Player 1 ever choose not to enter? And if Player 1 chooses to enter, why should Player 2 choose anything other than a Low Output. Remember this is a one shot game.

This process of starting with Player 2's decision and then looking at what is best for Player 1 given what is best for Player 2 is called backward induction and is the basis for the subgame perfect refinement to the Nash equilibrium. By using backward induction, we are able to eliminate all but {($s_E$ = 1, 1-$s_E$ = 0), ($s_L$ = 1, 1-$s_L$ = 0)} from the set of Nash equilibria.

But why? {($s_E$ = 0, 1-$s_E$ = 1), ($s_L$ =0, 1-$s_L$ = 1)} is a Nash equilibrium because if Player 2 played the high output when Player 1 entered, Player 1 would be worse off. The problem is so would Player 2. However, since Player 1 does not choose to enter in this equilibrium, Player 2's choice of a high output is not a problem because it is never put in a position to carry through with what would appear to be an irrational choice. When Player 1 chooses not to enter, Player 2's decision is off the equilibrium path. That is, in equilibrium, firm 2 never actually has to follow through with its choice, so it does not care whether or not it is rational. This type of irrational choice off the equilibrium path is referred to as an incredible threat. Subgame perfection eliminates any Nash equilibrium that is based on this type of incredible threat.

**Objective: Understand the subgame perfect equilibrium.**

Before defining the notion of a subgame perfect equilibrium, we must first define a subgame.

*Definition*

A subgame in an extensive form game:

(a) begins at a decision node *n* that is a singleton information set,
(b) includes all the decision and terminal nodes following *n* in the game tree (but no nodes that do not follow *n*), and
(c) does not cut any information sets (i.e. if a decision node *n*' follows *n* in the game tree, then all other nodes in the information set that contains *n*' must also follow *n* and must be included in the subgame).

A subgame can be thought of as a stand-alone game that is an endgame for a much larger game. For a decision node to be the start of a subgame, all players must know precisely when that node has been reached. Figure 2 and 3 offer some examples. In Figure 2, there are two subgames, one starts at player 2's decision node, the other at player 3's. In Figure 3, there are no subgames. Players 1, 2, and 3 know precisely where they are at beginning at player 2's and 3's decision node, but player 4 does not.

*Definition*

A Nash equilibrium is subgame perfect if the players' strategies constitute a Nash equilibrium in every subgame (Selten, 1965).

**Application: Stackelberg Duopoly**

Lets consider a Duopoly problem where there are 2 firms and demand equal to $P = a - q_1 - q_2$, where $q_1$ and $q_2$ are Firm 1's and 2's output. Firms have identical constant marginal costs $c$. If firms choose output simultaneously, the Cournot Nash equilibrium is $\{q_1^* = (a - c)/3, q_2^* = (a - c)/3\}$ or $Q^* = q_1^* + q_2^* = 2(a - c)/3$.

Now lets change the game slightly by assuming that Firm 1 gets to choose its output first and that Firm 2 gets to see this output before choosing its own. Instead of a simultaneously move game, we now have a dynamic game, so lets work backwards.

Before we start, we need to first think about what a strategy for Firm 1 and 2 is in this game. For Firm 1, a strategy is just an output choice, $q_1 \geq 0$. For Firm 2, a strategy is an output choice for each possible output choice Firm 1 could possibly make.

Firm 2's payoff is:
$U_2(q_2, q_1) = (a - q_1 - q_2)\, q_2 - cq_2 = (a - q_1 - c)\, q_2 - q_2^2$

Maximizing this payoff with respect to $q_2$ yields

FOC (for an interior solution):
$a - c - q_1 - 2\, q_2 = 0$, and

SOC:
$-2 < 0$, so we have a maximum.

Firm 2's best response function given Firm 1's output choice and its payoff maximizing strategy is $q_2^*(q_1) = (a - q_1 - c)/2$ for $a - c \geq q_1 \geq 0$, and 0 otherwise.

Firm 1's payoff is:
$U_1(q_1, q_2) = (a - q_1 - q_2)\, q_1 - cq_1 = (a - q_2 - c)q_1 - q_1^2$

Maximizing this payoff with respect to $q_2$ yields

FOC:
$(a - c - q_2) - (2 + \partial q_2/\partial q_1)q_1 = 0$

SOC:
$-(2 + \partial q_2/\partial q_1) - (\partial^2 q_2/\partial q_1^2)q_1 < 0$,

which is not guaranteed, but lets presume it is so.

Firm 1's payoff maximizing output is $q_1 = (a - c - q_2)/(2 + \partial q_2/\partial q_1)$.

Now for $q_2(q_1) = (a - q_1 - c)/2$, $\partial q_2/\partial q_1 = -1/2$, so
$q_1^* = (a - c - (a - q_1 - c)/2)/(2 - 1/2) \Rightarrow$
$q_1^* = (a - c)/2$.

$q_2^* = (a - (a - c)/2 - c)/2 \Rightarrow$
$q_2^* = (a - c)/4$.

$Q^* = 3(a - c)/4$.

In the Cournot Duopoly model the Nash equilibrium payoffs are:

$U_1^* = U_2^* = (a - c)^2/9$

In the Stackelberg model, the Nash equilibrium payoffs are:

$U_1^* = (a - c)^2/8$
$U_2^* = (a - c)^2/16$

Conclusions:
Stackelberg model has higher output and lower price.
Firm 1 has a higher payoff, while Firm 2 a lower payoff.
Total payoffs are lower.

Even though Firm 2 has better information on which to base its decision, it is worse off in the end because Firm 1 can dictate Firm 2's actions, at least to some extent.

As a final note, this Stackelberg equilibrium is the subgame perfect Nash equilibrium, but it is not the only Nash equilibrium. For example, you can construct a strategy for Firm 2 that results in the Cournot solution and a Nash equilibrium: $q_2 = (a - c)/3$ for all $q_1$. With this strategy, $q_2(q_1) = (a - c)/3$, $\partial q_2/\partial q_1 = 0$. Substituting into firm 1's FOC, yields $q_1 = (a - c)/3$. You can verify that no firm can do better by unilaterally changing its strategy, so $\{q_1 = (a - c)/3, q_2(q_1) = (a - c)/3\}$ is a Nash equilibrium.

This strategy is not subgame perfect however because if firm 1 plays $q_1 = (a - c)/2$ firm 2 setting $q_2 = (a - c)/3$ would not be a Nash equilibrium for the subgame starting with Firm 2's choice of output. Firm 2, could do better by changing its strategy. There is an infinite set of Nash equilibria in the Stackelberg game, but only one subgame perfect Nash equilibrium for our linear demand.

**Application: Stackelberg Rent Seeking**

Gordon Tullock is usually held responsible for spawning the rich literature on rent seeking. In short, suppose there is some prize to be had. The value of the prize to individual $i$ is denoted as $V_i$ where there are $N$ individuals. No prize can be had for free, so $x_i$ is the cost of effort expended in an attempt to win the prize. The probability that an individual wins the prize depends on how much effort everyone expends: $P_i(x_1, x_2, \ldots, x_N)$ where $\sum_{i=1}^{N} P_i(x_1, x_2, \ldots, x_N) = 1$.

Question: How much nonproductive effort will individuals waste trying to capture the prize? Will they spend more than the value of the prize?

The basic idea here is that there is a mechanism other than a market that is often used to determine the allocation of resources. Firms lobby politicians for access to public lands. Firms expend resources to seek patent protection, so they can monopolize an industry. Professors compete to be awarded lucrative research grants. The concern is that the cost of determining the allocation of these resources may ultimately exceed their value.

But all this is really an aside to the point I really want to make. In the Stackelberg duopoly model having more information on which to base your decision is a bad thing because the player with less information has some control over your response, but is this always the case?

Let $N = 2$ and $P_i(x_i, x_j) = x_i/(x_i + x_j)$ to keep things easy.

Player *i*'s expected payoff can then be written as:

$U_i = V_i x_i/( x_i + x_j) - x_i$

FOC (for an interior solution):
$V_i x_j/( x_i + x_j)^2 – 1 = 0$

SOC:
$-2V_i x_j/( x_i + x_j)^3 < 0.$

Player *i*'s best response function is then $x_i = \begin{cases} \sqrt{V_i x_j} - x_j & for\ 0.0 < x_j \le V_i \\ 0, & otherwise \end{cases}$.

Solving the system of players' best response functions leads to the Cournot Nash equilibrium for the rent seeking game: $x_i{*} = V_i \frac{V_i V_j}{(V_i + V_j)^2}$. Total effort expended is

$x_1{*} + x_2{*} = \frac{V_1 V_2}{V_1 + V_2}$, which equals ½ of the value of the prize if both players value it

equally. Player *i*'s expected payoff is $U_i{*} = V_i \left\{ \frac{V_i^2}{(V_i + V_j)^2} \right\}$.

According to this model total effort will never exceed the value of the prize to either individual.

But who has the advantage if one player gets to move first?

Suppose Player 1 gets to move first. From above, we know that Player 2's Nash equilibrium strategy given Player 1's effort is $x_2 = \begin{cases} \sqrt{V_2 x_1} - x_1 & \textit{for } 0.0 < x_1 \le V_2 \\ 0, & \textit{otherwise} \end{cases}$. So assuming an interior solution, player 1's payoff given this strategy is

$$U_1 = V_1 \frac{x_1}{\sqrt{V_2 x_1}} - x_1 = V_1 \sqrt{\frac{x_1}{V_2}} - x_1 .$$

Maximizing with respect to $x_1$ yields:

FOC (interior):

$$\frac{V_1}{2}(V_2 x_1)^{-\frac{1}{2}} - 1 = 0$$

SOC:

$$-\frac{V_1}{4} V_2^{-\frac{1}{2}} x_1^{-\frac{3}{2}} < 0$$

So, $x_1{*} = V_1 \frac{V_1}{4V_2}$, $x_2{*} = \frac{V_1}{2} - \frac{V_1^{\,2}}{4V_2}$, $x_1{*} + x_2{*} = \frac{V_1}{2}$, $U_1 = V_1 \frac{V_1}{4V_2}$, and $U_2 = V_2 - V_1 + \frac{V_1^{\,2}}{4V_2}$.

Conclusions: Comparing expected payoffs for the simultaneous move Cournot Nash equilibrium to the expected payoff for the subgame perfect Stackelberg equilibrium shows that Players 1's payoff always has to be higher in the Stackelberg equilibrium. This makes intuitive sense because the player that moves first can always choose the Cournot Nash effort to which the second mover's best response is the Cournot Nash effort. So to change this strategy, it must be better off. For Player 2, the player with more information, the result is a little messier. When $V_2 > V_1 > 0$, Player 2 is better off if Player 1 lets him know what is going on before hand. When $2V_2 > V_1 > V_2$, Player 2 is better off not knowing what Player 1 does. So contrary to the duopoly model, a Player can be better off knowing what an opponent will do in advance.

In these types of games there are two types of effects: the timing effect and the information effect. These two effects may reinforce each other or counteract each other depending on the structure of the game.

**Objective: Understand the empirical weaknesses of the subgame perfect equilibrium.**

The predictive power of the subgame perfect equilibrium has been challenged empirically on two fronts.

First, when threats are not too costly, they are frequently carried out. Figure 4 shows results reported in Goeree and Holt (2001) for two different parameterizations of the entry deterrence game. The change in payoffs does not change the equilibrium prediction, but does substantially change observed behavior. When carrying out a threat costs Player 2 40 it is never done, but when it only costs 2 it is done nearly half the time. Furthermore, about 20% of the Player 1s recognized the importance of this difference when making their decisions. Explanations of this behavior have focused on some notion of spite or revenge.

Another classic example of the predictive failure of the subgame perfect equilibrium is the centipede game illustrated in Figure 5. Figure 5 shows two versions and the percentage of observed behavior reported in McKelvey and Palfrey (1992).

The game goes like this, a player starts by choosing either to Take the current distribution of payoffs that is in his favor or Pass to the next player. The players' total payoffs increase with every pass, but the distribution of payoffs is less favorable unless a player chooses Take. Players alternate with this Take/Pass decision until the end of the game.

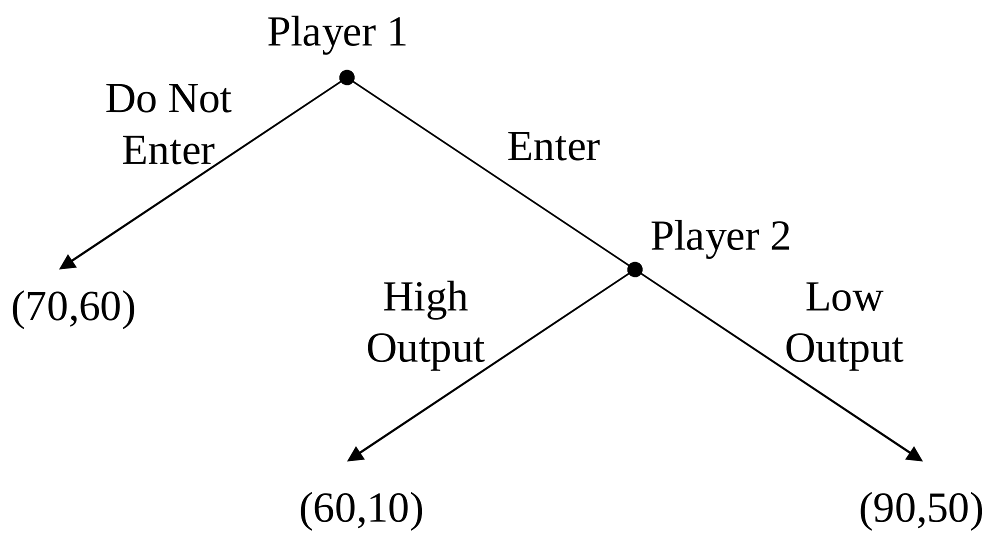
What is the subgame perfect equilibrium for this game? Working backward from the end of the game, the player with the final decision is always better off choosing Take, but if the last player chooses Take, the second to last player's best response is also Take, and so on. The subgame perfect solution for the game is for the first player to choose Take.

The percentage of observed behavior reported in Figure 5 shows that this is certainly not the case. Why? McKelvey and Palfrey offer an explanation that transforms this complete information game into an incomplete information game by assuming there are some players who are altruist (i.e. they like others to earn higher payoffs as well as themselves).

The failure of subgame perfection in the entry deterrence game is more disturbing to me than in the Centipede game because Player 2 is unequivocally worse off in the end. In the Centipede game, a person is often better off by choosing to Pass early in the game. Furthermore, choosing Pass could send a signal that you are willing to cooperate to everyone's advantage, so others may also be more willing to cooperate. Of course, this cooperation is tenuous in the end because somebody ends up on the short end of the jackpot. But even on the short end of the jackpot a player is often better off than if he played according to the subgame perfect equilibrium.

Too summarize, when carrying out a threat is not too costly, it should be believed. Also, when a subgame perfect equilibrium leads to inferior payoffs, people often ignore its logic and bet that other players will too.

Figure 1: Entry Deterrence Game




(Player 1's Payoff, Player 2's Payoff)

Figure 2: Example Subgames

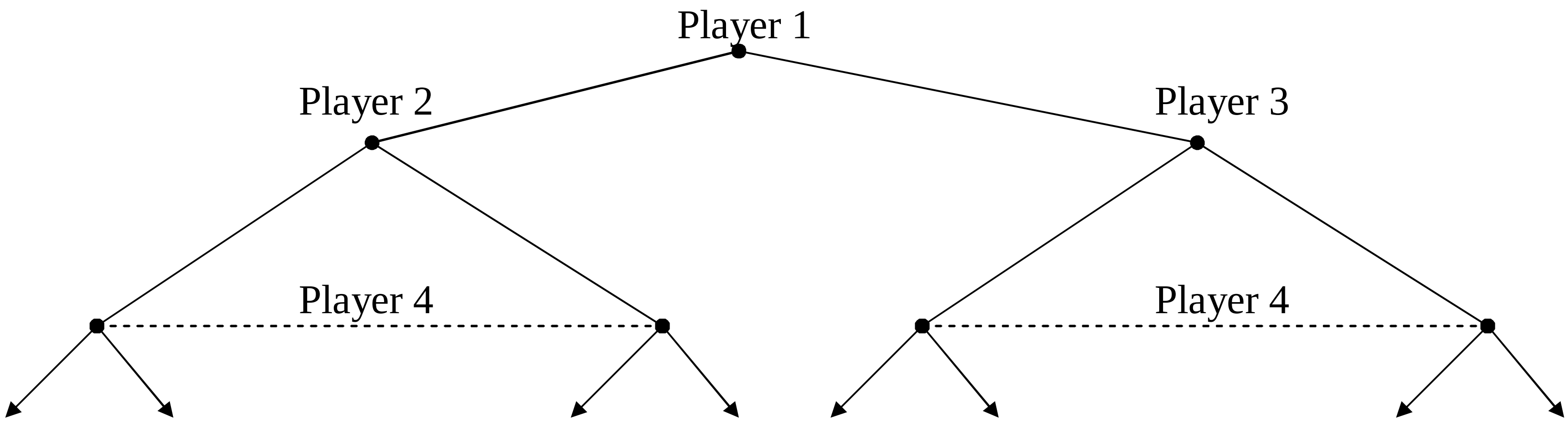


Figure 3: Example Of No Subgames

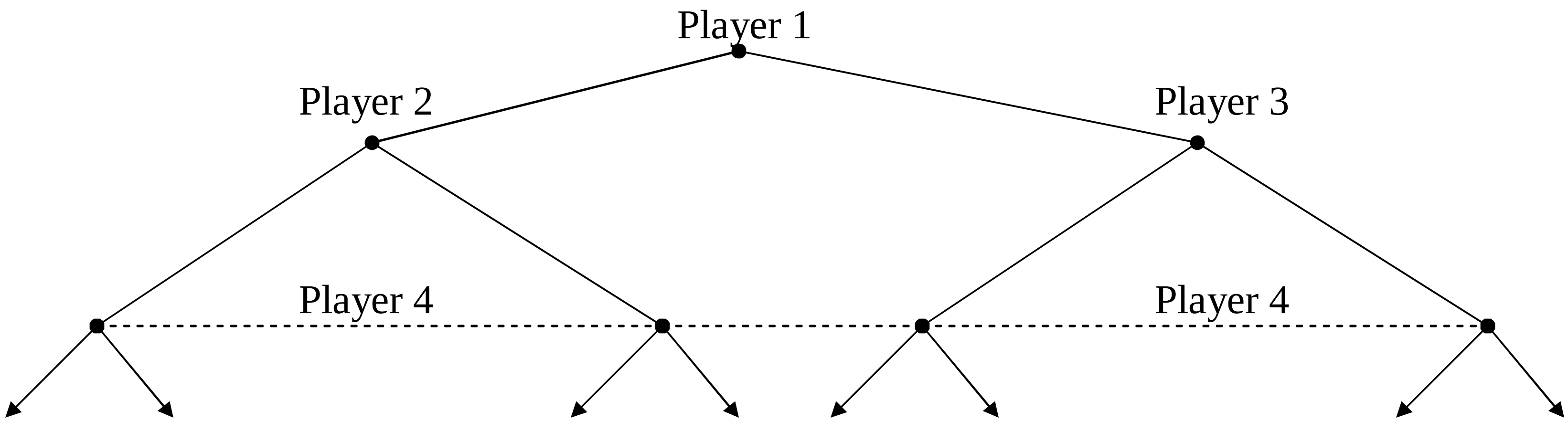
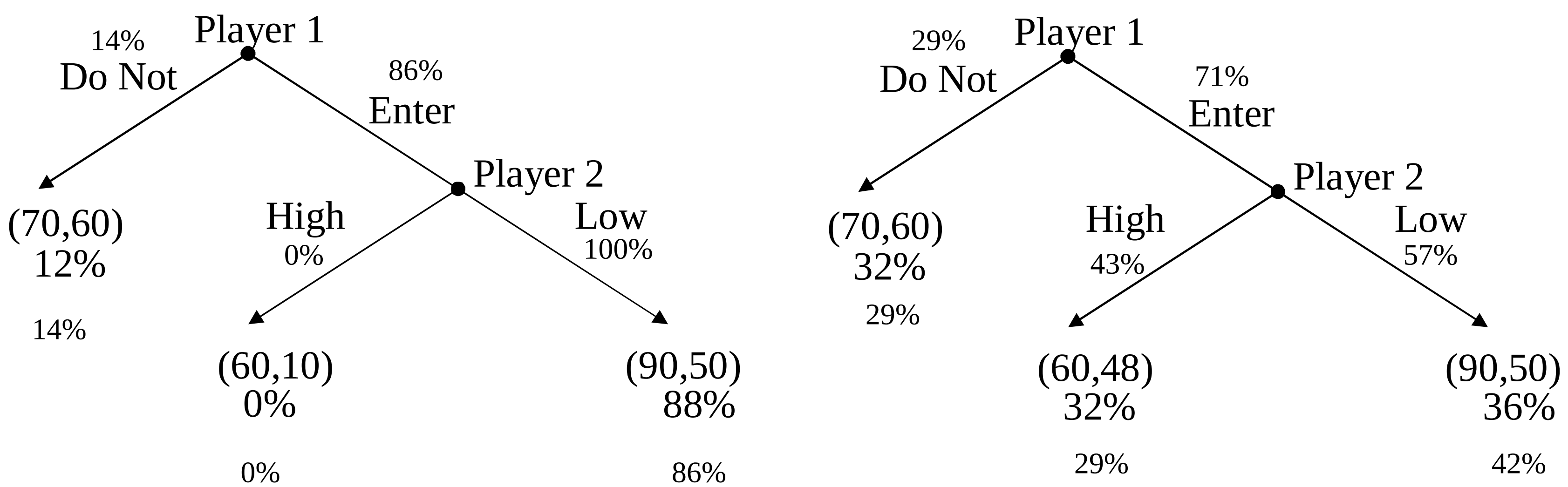
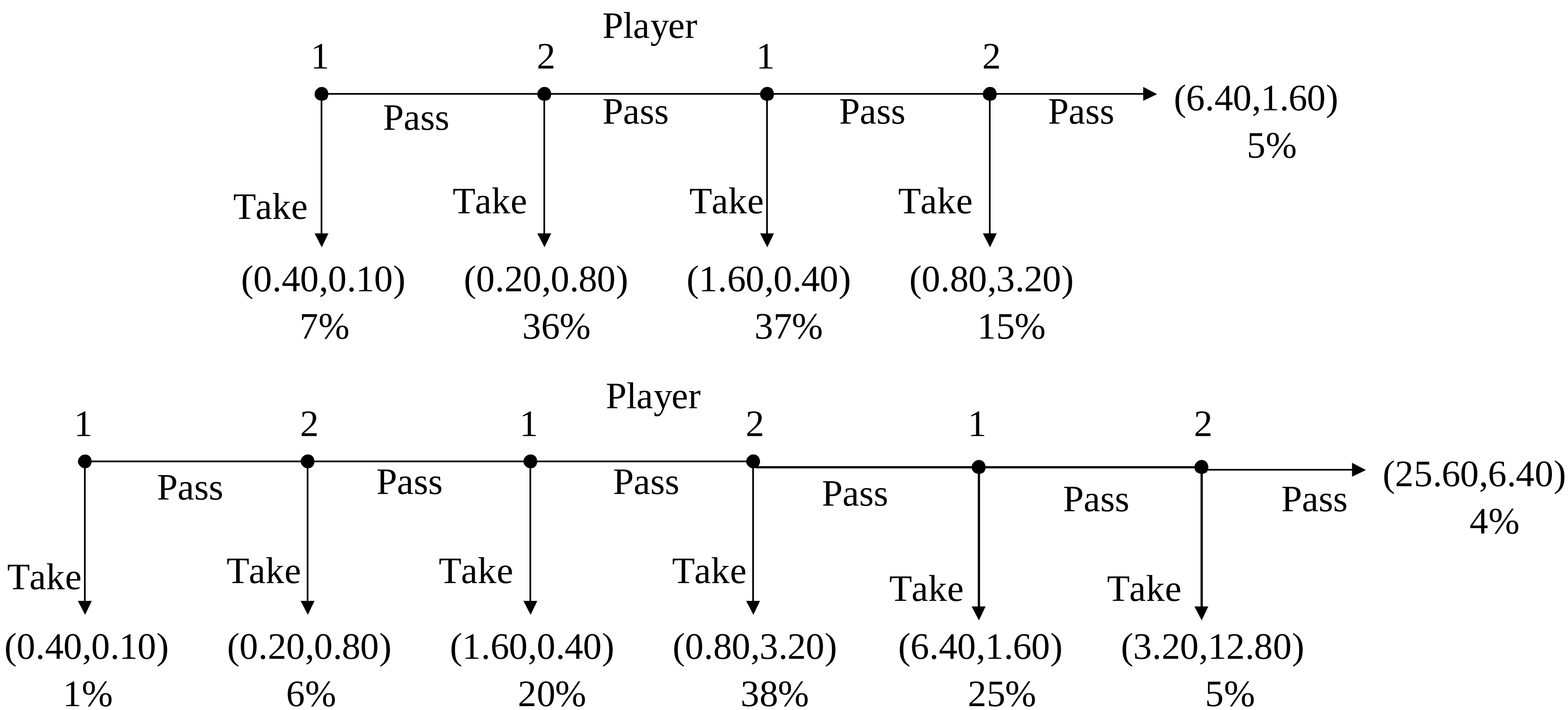


Figure 4: Empirical Evidence From The Entry Deterrence Game




(Player 1’s Payoff, Player 2’s Payoff)
Percentage of Observed Play

Figure 5: The Centipede Game




(Player 1's Payoff, Player 2's Payoff)
Percentage of Observed Play