Static Games of Complete Information

I. Introductory Comments

A. Strategic interaction.

1. Game theory analyzes situations in which each player's optimal choice depends upon the choices of his/her fellow players. "Strategic interaction."
2. Game theory a much more difficult problem to solve than engineering problem or perfect competition (environment does not fight back!).
3. Since the choices of others, player must have a way to predict what others will do.

B. Assumptions and warnings.

1. Completely specify the "rules of the game." (Players, possible choices, sequences of moves, information structure, and payoffs). Rules of the game are **common knowledge**. All players know the rules of the game. Further, all players know that all players know the rules of the game. And all players know that all players know that all players know the rules of the game….(example: imperfectly reliable email exchange)
2. Assume "rationality": players maximize expected utility. Note: game theory requires more rationality than traditional decision theory. Must predict other players' choices, which depends on understanding their expected utility maximization problems, which, of course, depends on their view of other player's expected utility maximization problems, which, depends upon…
3. Normative model: what players *should* do. Rational players who understand the rules of the game should play according to game theory in order to maximize their expected utility. Experiments show that real players may deviate from predictions of game theory. Possible reasons:
   a. Players fail to fully understand the rules of the game. Players may have to learn to play the game before their behavior settles on predicted behavior.
   b. Players may not be expected utility maximizers. Example: experiments on **framing** by Kahneman and Tversky:
      Choose: a) $100 with probability 1 or b) 0 with probability .75, $400 with probability .25. (Most choose a). I give you $400. Now choose: c) lose $300 with probability 1, or d) lose 0 with probability .25 and lose $400 with probability .75. (Most people choose d). Expected utility maximizers if choose (a) must choose (c), if choose (b), must choose (d).
   c. Players may not correctly anticipate other players' choices – either failure of logic or belief that other players are not fully rational.
   d. In experiments, payoffs are usually money payoffs. But game theory is about utility payoffs. Role of emotions (what if fair, reciprocity…). Translation from money payoff to utility payoffs – need to understand players psychology.

II. Normal Form Games (Static Games of Complete Information)

A. Notation: rules of the game

1. Players: $i = 1, 2, \ldots, n;\ i \in N$
2. Strategies: s_i is a pure strategy (action) for player i, $s_i \in S_i$ is set of possible pure strategies for player i.
   $s = (s_1, s_2, \ldots, s_n)$, $s \in S$ where $S = X\, S_i$
3. Utility (preferences): $U_i(s)$, describes player i's ranking of various possible outcomes. $U = \{U_1, U_2, \ldots, U_n\}$

{*N, S, U*} are the rules of the game, which are common knowledge.

B. Solution concepts:

1. How should we predict what players will do? What strategy combinations are likely to be chosen?
2. There is no single right answer. Game theorists have settled on some solution concepts – seem reasonable or have desirable properties. Analogy to econometrics: why is least-squares chosen as the estimator of choice? (Good properties)
3. Solution concepts for static games of complete information:
   a. Nash equilibrium
   b. Dominant strategy equilibrium
   c. Iterated strict dominance
   d. Maxi-min solution

C. Nash equilibrium

1. Core concept of game theory. Most applications in economics use Nash equilibrium or refinement of Nash equilibrium.
2. Let s_i represent the strategy of player $i \in N$ and let $s_{\sim i}$ represent the strategies of all players $j \neq i \in N$
3. Definition 1: $s^* \in S$ is a **pure strategy Nash equilibrium** if for all players $i \in N$, $U_i(s_i^*, s_{\sim i}^*) \geq U_i(s_i, s_{\sim i}^*)$ for all $s_i \in S_i$. {No profitable unilateral deviations}
4. Definition 2:
   a. A **best response** function for player i:
      $br_i(s) = \{s_i \in S_i: U_i(s_i, s_{\sim i}) \geq U_i(s_i', s_{\sim i})$ for all $s_i' \in S_i\}$
   b. Best response correspondence:
      $br(s) = X\, br_i(s)$
   c. $s \in S$ is a pure strategy Nash equilibrium if $s \in br(s)$ {strategy is a best response to itself}
5. Note: definition 2 is useful for existence proofs. Nash equilibrium is a fixed point of best response correspondence. However, existence is not usually the problem with Nash equilibrium. Multiplicity of equilibria is more often the problem.

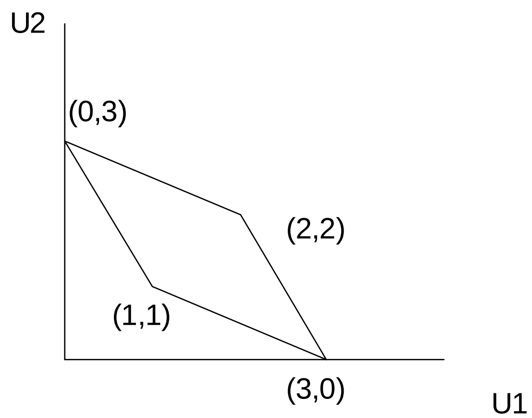
D. Simple examples

1. Prisoner's Dilemma

Player 2 Player 1	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1

a. Unique Nash equilibrium is for both players to defect (D,D): denote Nash equilibrium by ***strategies***, not by payoff.

Note: (D,D) is only pure strategy combination that is not Pareto optimal. Game theory solutions do not always satisfy fundamental welfare theorems. {Why not? – externality, or non-price taking behavior}



2. Battle of the Sexes

Player 2 Player 1	Football	Ballet
Football	2, 1	0, 0
Ballet	0, 0	1, 2

a. Two pure strategy Nash equilibria: (F, F), (B, B).
b. General problem: with multiple equilibria, what is the prediction of the game? How do we know that players will even get to equilibrium?
c. Need of refinement?
    i. Schelling: focal points. Meet in NYC tomorrow. Grand Central Station at noon.
    ii. Pareto dominance: may work.

Player 2 Player 1	Left	Right
Up	2, 2	0, 0
Down	0, 0	1, 1

But, it may not.

Player 2 Player 1	Left	Right
Up	9, 9	0, 8
Down	8, 0	7, 7

3. Matching Pennies

Player 2 Player 1	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

a. There does not exist a pure strategy Nash equilibrium.
b. There does exist a mixed strategy Nash equilibrium {How would you play this game?}. Will show an existence proof once introduced mixed strategies.

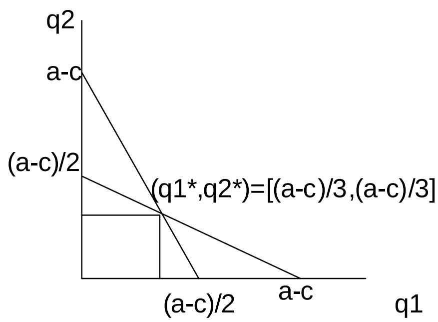
E. Economic examples

1. Cournot Oligopoly

a. Duopoly with linear demand, constant marginal cost
Players: $i = 1, 2$
Strategies: $q_i \geq 0$
Payoffs: (profit) $U_i(q_1, q_2) = (a - q_1 - q_2)\, q_i - c\, q_i$

Best response function (FOC): $a - 2q_i - q_{\sim i} - c = 0$ or $q_i\,(q_{\sim i}) = (a - c - q_{\sim i})/2$

Nash equilibrium (must satisfy both best response functions; 2 linear equations in 2 unknowns): $q_1^* = q_2^* = (a - c)/3$



Note: convincing once you are there that you would stay there, but how to get there in the first place? Adjustment to other players strategy; but game is static.

b. N-firm oligopoly: $i = 1, 2, \ldots, n$
Best response function: $a - 2q_i - q_{\sim i} - c = 0$, or $a - q_i - Q - c = 0$
Sum over n-firms: $n(a - c) - (n+1)Q = 0$; $Q = \dfrac{n(a-c)}{n+1}$

$$q_i = \frac{(a-c)}{n+1}$$

Note: as n goes to infinity, $Q = (a - c)$, and $P = c$; competitive result.

c. N-firm oligopoly with general demand and cost
$U_i(q_i, q_{\sim i}) = \mathrm{P}(q_i, q_{\sim i})\, q_i - C(q_i)$

Best response function: $\mathrm{P}'(q_i, q_{\sim i})\, q_i + \mathrm{P}(q_i, q_{\sim i}) - \mathrm{C}'(q_i) = 0$
N non-linear equations in N unknowns: no guarantee of existence or uniqueness. See Tirole p. 224- 226 for sufficient conditions for existence and uniqueness of Nash equilibrium in a Cournot game.

2. Common property resource

Players: harvesters: $i = 1, 2, \ldots$, n.
Strategies: how many cows to put on the commons: $q_i \geq 0$. Let Q = sum qi
Payoffs: total weight of cows on commons: $aQ - Q^2$ for $0 \leq Q \leq a$, 0 otherwise.

$U_i(q_i, q_{\sim i}) = p\,(aQ - Q^2)\, q_i/Q - c\, q_i$
$= p\,(a - Q)\, q_i - c\, q_i$

Best response function: $p(a - 2q_i - q_{\sim i}) - c = 0$ or $(a - q_i - Q) - c/p = 0$
Sum over all N harvesters: $na - (n+1)Q - n\, c/p = 0$

$Q = n(a\text{-}c/p)/(n+1)$

Use this to solve for q_i: $q_i = (a\text{-}c/p)/(n+1)$
What happens as n goes to infinity? $Q = a\text{-}c/p$; $U_i(q_i, q_{\sim i}) = 0$.

Comparison with oligopoly: as n goes to infinity in oligopoly, get perfect competition (efficiency). Here as n goes to infinity you get the tragedy of the commons- total rent dissipation. Efficiency occurs for $n = 1$.

F. Dominant Strategy Equilibrium

1. Motivation
   a. Nash equilibrium is the dominant equilibrium concept used to make predictions about games. But other solution concepts exist as well.
   b. Some situations may be able to appeal to other concepts and make story more convincing about prediction.
   c. One such concept is dominant strategy equilibrium: *if* it exists it is usually fairly compelling as a prediction.

2. Definition
   a. Dominant strategy: for $s_i, t_i \in S_i$,
      i. s_i weakly dominates t_i if: $U_i(s_i, s_{\sim i}) \geq U_i(t_i, s_{\sim i})$ for all $s_{\sim i} \in S_{\sim i}$;
      ii. s_i dominates t_i if: $U_i(s_i, s_{\sim i}) \geq U_i(t_i, s_{\sim i})$ for all $s_{\sim i} \in S_{\sim i}$ and $U_i(s_i, s_{\sim i}) > U_i(t_i, s_{\sim i})$ for some $s_{\sim i} \in S_{\sim i}$
      iii. s_i strictly dominates t_i if: $U_i(s_i, s_{\sim i}) > U_i(t_i, s_{\sim i})$ for all $s_{\sim i} \in S_{\sim i}$.
   b. Dominant strategy equilibrium: $s \in S$ is a (weakly/strictly) dominant strategy equilibrium if for all $i \in N$, and all $t_i \in S_i$, s_i (weakly/strictly) dominates t_i.

3. Example: Prisoner's Dilemma

Player 2 Player 1	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1

(D, D) is a dominant strategy equilibrium.

4. Second price sealed bid auctions (strictly speaking, this game is one of incomplete information – but it is a rare economic situation with a dominant strategy equilibrium).
   a. N players i = 1, 2, …, N. Each player has a value for a good to be auctioned ($v_1, v_2, \ldots, v_N$) known only by the player
   b. Strategies: players bid simultaneously ($b_1, b_2, \ldots, b_N$).
   c. Payoffs: High bid wins the auction, pays an amount equal to the second highest bid.
   d. Analysis: consider player *i*, let $h_{\sim i}$ be the highest bid for all players not *i*.

      Payoff for player *i*: $U_i = \begin{cases} v_i - h_{\sim i} \text{ for } b_i > h_{\sim i} \\ 0 \text{ for } b_i < h_{\sim i} \end{cases}$ (ignore ties)
   e. Claim: dominant strategy to bid the value: $b_i = v_i$. If $b_i > h_{\sim i}$ then win the auction and get $v_i - h_{\sim i} > 0$. If $b_i < h_{\sim i}$ then don't win and get 0.
   f. Suppose $b_i > v_i$. If $b_i > v_i$ then there is some $h_{\sim i} : b_i > h_{\sim i} > v_i$. In this case "win" auction and get payoff of $v_i - h_{\sim i} < 0$. If $b_i = v_i$ would have gotten 0 in this case. For $h_{\sim i}$ outside of this range, get same payoffs as when set bid equal to value.
   g. Suppose $b_i < v_i$ If $b_i < v_i$ then there is some $h_{\sim i} : v_i > h_{\sim i} > b_i$. In this case don't win the auction and get payoff of 0. If $b_i = v_i$ would have won the auction and gotten $v_i - h_{\sim i} > 0$. For $h_{\sim i}$ outside of this range, get same payoffs as when set bid equal to value.
   h. Appeal of second price auction: "truth-telling" is a dominant strategy. Not the case in first price auction where want to lower bid to reduce payment. Here payment is not tied to own bid.

5. Dominant strategy equilibrium typically fail to exist. Not many cases in economics where dominant strategies exist.

G. Iterated Strict Dominance

1. Dominant strategies may fail to exist but might use the idea of dominance to generate a prediction.

2. No rational player should ever play a strictly dominated strategy. Reasonable prediction of play would throw out dominated strategies. But then more strategies may become strictly dominated. Continue to toss out strictly dominated strategies until process ceases: iterated dominance.
3. Only discard strategies that are strictly dominated: $U_i(s_i, s_{\sim i}) > U_i(t_i, s_{\sim i})$ for all $s_{\sim i} \in S_{\sim i}$.
4. If process of tossing out strictly dominated strategies results in a single outcome, we say the game is solvable by iterated strict dominance.

5. Example:

	L	C	R
U	4, 3	5, 1	6, 2
M	2, 1	8, 4	3, 6
D	3, 0	9, 6	2, 8

C is strictly dominated by R for player 2.

	L	R
U	4, 3	6, 2
M	2, 1	3, 6
D	3, 0	2, 8

M and D are strictly dominated by U.

	L	R
U	4, 3	6, 2

R is strictly dominated by L.
Iterated strict dominance prediction: (U, L)

6. Is iterated strict dominance as "reasonable" as strict dominance? Not really. It relies on a chain of logic that depends upon the reasonableness of play of the rival player. What if the other player is capable of making "mistakes?"
7. Example:

	L	R
U	8, 10	-100, 9
D	7, 6	6, 5

What would you predict player 1 to do? (Note: U, L is iterated strict dominance prediction and Nash equilibrium)
8. If players have limited reasoning ability, perhaps because they are in a novel situation or because the game is complex, it might not be reasonable to assume that they can apply a complex iterated chain of logic. Even if others are

sophisticated, if there are doubts about the sophistication of rivals this might be enough for players to chose not to apply iterated dominance. Example: Beauty pageant game. Experimental evidence: players use limited steps of iterated reasoning: mean number of iterations is 2 (but can learn to do more).

9. Iterated strict dominance may fail to yield a prediction (Ex: Battle of the Sexes).
10. Closely related notion of rationalizability (Bernheim 1984, Pearce 1984). What are all the strategies that a rational player could play? Rule out those strategies that are not a best response to beliefs about rival's strategies. Restrict beliefs to be rational in the sense that there should be zero weight on rival's strategies that are not best responses. And, this may lead to further refining of what is a rational play… In two person games, rationalizability is the same as iterated strict dominance. In N person games, these concepts need not be the same.

H. Maxi-min Solution

1. If fearful that rival players might make mistakes, may wish to be cautious in how you play – avoid bad outcomes - idea behind maxi-min strategy. (Or if paranoid – they are out to get me…)
2. Strategy s_i* is a maximin strategy if it maximizes player *i*'s minimum possible payoff: $s_i^* = \max_{s_i} [\min_{s_{\sim i}} U(s_i, s_{\sim i})]$
3. Maximin solution: each player plays a maximin strategy.
4. Example

	L	R
U	0,0	3,1
D	4,3	1,2

Maximin solution is (D,R)

6. Maximin solution prominent when game theorists primarily studied zero-sum games. Zero-sum games (constant sum games) are games of pure competition – one persons gain is another loss. Example: matching pennies. In zero-sum games, paranoia is justified – they are out to get you (so they get more).
7. In non-zero sum games, maxi-min may not make much sense. Example:

	L	R
U	0,0	3,1
D	400,300	100,2

Columns player should be fairly certain that rows player will play D, in which case the column player can get 300 by playing L.

I. Mixed Strategy Nash Equilibrium

1. Definition of Mixed Strategies

$s_i(s_i)$ = probability that player *i* will play strategy s_i.

$s_i(S_i)$ is a mixed strategy (tells the probability of playing each pure strategy)

Σ_i is the set of mixed strategies for player *i*

$$\Sigma_i = \{ s_i(S_i) : \sum_{s_i \in S_i} s_i(s_i) = 1, s_i(s_i) \geq 0, \text{for all } s_i \in S_i \}$$

2. Mixed Strategy Nash Equilibrium
   a. Can also define other equilibrium (dominant strategy equilibrium, iterated dominance) with mixed strategies just as you can with pure strategies
   b. Definition: A mixed strategy profile s^* is a Nash equilibrium if, for all players $i \in N$, $U_i(s_i^*, s_{\sim i}^*) \geq U_i(s_i, s_{\sim i}^*)$ for all $s_i \in S_i$.

3. Example: Battle of the Sexes

		Wife (*W*) Football: *F*	Wife (*W*) Ballet: *B*
Husband (*H*)	Football: *F*	2, 1	0, 0
	Ballet: *B*	0, 0	1, 2

Recall that in this game there were two pure strategy Nash equilibria, but what about a mixed strategy Nash equilibrium.

The wife prefers *F* (*B*) when
$U_W(F) >(<) U_W(B) \Rightarrow$
$1 s_H(F) + 0(1 - s_H(F)) >(<) 0 s_H(F) + 2(1 - s_H(F)) \Rightarrow$
$s_H(F) >(<) 2/3$, so the wife's best response function can be written as

$$br_W(s) = \begin{cases} s_W(F) = 0, & \textit{for } 2/3 > s_H(F) \geq 0.0 \\ s_W(F) \in [0,1], & \textit{for } s_H(F) = 2/3 \\ s_W(F) = 1, & \textit{for } 1.0 \geq s_H(F) > 2/3 \end{cases}.$$

The husband prefers *F* (*B*) when
$U_H(F) >(<) U_H(B) \Rightarrow$
$0 s_W(F) + 2(1 - s_W(F)) >(<) 1 s_W(F) + 0(1 - s_W(F)) \Rightarrow$
$2/3 >(<) s_W(F)$, so the husband's best response function can be written as

$$br_H(s) = \begin{cases} s_H(F) = 0, & \textit{for } 1/3 > s_W(F) \geq 0.0 \\ s_H(F) \in [0,1], & \textit{for } s_W(F) = 1/3 \\ s_H(F) = 1, & \textit{for } 1.0 \geq s_W(F) > 1/3 \end{cases}.$$

Figure below illustrates the best response functions and shows that there is indeed a mixed strategy Nash equilibrium where the wife chooses football with probability 1/3 and the husband chooses football with probability 2/3 (as well as two pure strategy equilibria).

4. Existence Theorem for finite games:
   Every finite game has a Nash equilibrium in the game with mixed strategies. (Nash 1950)

   Sketch of Proof: Application of a fixed point theorem. A Nash equilibrium is a fixed point of the best response correspondence: $s \in br(s)$.

   Kakutani's Fixed Point Theorem: sufficient conditions for a correspondence $br(s)$ to have a fixed point are for $s \in \Sigma$:

   i) Σ is a compact, convex and nonempty subset of a (finite-dimensional) Euclidean space
   ii) $br(s)$ is nonempty for all σ.
   iii) $br(s)$ is convex for all σ.
   iv) $br(s)$ is upper hemi-continuous ("continuous").

   Not difficult to show that each condition is satisfied: see Fudenberg and Tirole p. 29-30.
   (i) player i's strategy space Σ_i is compact, convex and non-empty
   $[\Sigma_i = \{s_i(S_i): \sum_{s_i \in S_i} s_i(s_i) = 1, s_i(s_i) \geq 0, \text{for all } s_i \in S_i]$ so that Σ is also.
   (ii) A continuous correspondence obtains a maximum (minimum) over a compact space so a best response exists for any given mixed strategy.
   (iii) Expected utility is linear in probabilities so that convexity of best response is assured.
   (iv) Expected utility is continuous in probabilities.

   Note: mixed strategies are essential for proof because otherwise the strategy set is not convex.

   Intuition: 2x2 game: player 1 best response goes from left hand side to right hand side of box. Player 2 best response goes from bottom to top. Given that both of these are continuous, they must cross at least once.

5. Existence Theorem for continuous games: When S_i are nonempty, compact, convex subsets of Euclidean space, payoffs U_i are continuous in S and quasiconcave in S_i, there exists a pure strategy Nash equilibrium. (Debreu 1952, Glicksburg 1952, Fan 1952)