

**Applied Game Theory**  
APEC 8205

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**Problem Set #3 Answers**

1. Consider the two player game where Player 1 can choose either *Up* or *Down* and Player 2 can choose either *Left* or *Right*. Suppose that Player 1 is either cooperative with probability  $P$  or uncooperative with probability  $1 - P$ . Player 1 knows whether or not she is cooperative, but Player 2 doesn't. The tables below show the players payoffs for the game when Player 1 is cooperative and when she is uncooperative.

- a) What are the pure strategy Bayesian Nash equilibria for this game, if any?  
b) Characterize the mixed strategy Bayesian Nash equilibria for this game.

|          |             | Player 2                                      |              |    |    |  |              |
|----------|-------------|---|--------------|----|----|--|--------------|
|          |             | Payoffs with a<br>Cooperative Player 1<br>$P$ |              |    |    | Payoffs with an<br>Uncooperative Player 1<br>$1 - P$ |              |
|          |             | <i>Left</i>                                   | <i>Right</i> |    |    | <i>Left</i>  | <i>Right</i> |
| Player 1 | <i>Up</i>   | 1   | -1           | -1 | -1 |  |              |
|          | <i>Down</i> | -1  | 1            | 1  | 1  |  |              |

**ANSWER:**

a) Let  $\sigma$  be the probability Player 2 chooses *Left* and  $1 - \sigma$  be the probability she chooses *Right*. Let  $\sigma_C$  be the probability a cooperative Player 1 chooses *Up* and  $1 - \sigma_C$  be the probability she chooses *Down*. Let  $\sigma_U$  be the probability an uncooperative Player 1 chooses *Up* and  $1 - \sigma_U$  be the probability she chooses *Down*. A cooperative Player 1's expected payoff from choosing *Up* is  $\pi_C(Up) = \sigma - (1 - \sigma)$ . Choosing *Down* yields  $\pi_C(Down) = -\sigma + (1 - \sigma)$ . Therefore, the best

$$\text{response for a cooperative Player 1 is } s_C(s) = \begin{cases} 1, & s > 1/2 \\ [0,1], & s = 1/2 \\ 0, & s < 1/2 \end{cases} . \text{ An}$$

uncooperative Player 1's expected payoff from choosing *Up* is  $\pi_U(Up) = -\sigma + (1 - \sigma)$ . Choosing *Down* yields  $\pi_U(Down) = \sigma - (1 - \sigma)$ . Therefore, the best response

$$\text{for an uncooperative Player 1 is } s_U(s) = \begin{cases} 1, & s < 1/2 \\ [0,1], & s = 1/2 \\ 0, & s > 1/2 \end{cases} . \text{ Player 2's expected}$$

payoff from choosing *Left* is  $\pi(Left) = (\sigma_C - (1 - \sigma_C))P + (\sigma_U - (1 - \sigma_U))(1 - P) =$

$2P(\sigma_C - \sigma_U) + 2\sigma_U - 1$ . Choosing *Right* yields  $\pi(\text{Right}) = (-\sigma_C + (1 - \sigma_C))P + (-\sigma_U + (1 - \sigma_U))(1 - P) = 2P(\sigma_U - \sigma_C) + 1 - 2\sigma_U$ . Therefore, the best response for Player 2 is  $s(s_C, s_U) = \begin{cases} 1, & P s_C + (1 - P) s_U > 1/2 \\ [0,1], & P s_C + (1 - P) s_U = 1/2 \\ 0, & P s_C + (1 - P) s_U < 1/2 \end{cases}$ .

There are 4 possible pure strategies for Player 1:  $(Up, Up)$ ,  $(Up, Down)$ ,  $(Down, Up)$ , and  $(Down, Down)$  where the first element corresponds to a cooperative Player 1 and the second corresponds to an uncooperative one. Two of these,  $(Up, Up)$  and  $(Down, Down)$ , we can eliminate right away because the  $\sigma$  that makes a cooperative type want to choose *Up* (*Down*) is different from the  $\sigma$  that makes an uncooperative type want to choose *Up* (*Down*). However,  $(Up, Down)$  is a possible equilibrium strategy if  $\sigma = 1$  and  $(Down, Up)$  is a possibility for  $\sigma = 0$ . Now, if Player 1 chooses  $(Up, Down)$ ,  $\sigma(1,0) = 1$  is a best response for Player 2 only if  $P\sigma_C + (1 - P)\sigma_U = P > 1/2$ . If Player 1 chooses  $(Down, Up)$ ,  $\sigma(0,1) = 0$  is a best response for Player 2 only if  $P\sigma_C + (1 - P)\sigma_U = 1 - P < 1/2$  or  $P > 1/2$ . Therefore, we have two possible pure strategy Bayesian Nash equilibria provided  $P > 1/2$ :  $\{(Up, Down), Left\}$  and  $\{(Down, Up), Right\}$ . If  $P < 1/2$ , there are no pure strategy Bayesian Nash equilibria.

- b) From above we know that for a cooperative Player 1 to want to mix  $\sigma = 1/2$ . The same is true for an uncooperative Player 2. For  $\sigma = 1/2$  to be a best response for Player 2,  $P\sigma_C + (1 - P)\sigma_U = 1/2$  or  $s_C = \frac{1}{2P} - \frac{1 - P}{P} s_U$ . Since  $1 \geq \sigma_C \geq 0$ ,  $\frac{1}{2(1 - P)} \geq s_U \geq \frac{1 - 2P}{2(1 - P)}$ . Therefore, we have the set of mixed strategy Bayesian Nash equilibria  $\left\{ \left( \left( \frac{1}{2P} - \frac{1 - P}{P} s_U, s_U \right), \frac{1}{2} \right) \right\}$  for  $\frac{1}{2(1 - P)} \geq s_U \geq \frac{1 - 2P}{2(1 - P)}$ .

2. Suppose we have two fishermen. Each expends effort  $x_i$  for  $i = 1, 2$  to catch fish. The total catch,  $F$ , is a quadratic function of total effort:  $F(X) = 2(\alpha^2 X - 0.5\beta X^2)$  where  $X = x_1 + x_2$  and each fisherman shares in the catch equally. The price of fish is normalized to 1, while the cost of effort is  $C(x_i) = 0.5x_i^2$ . Finally, suppose fisherman 1 is experienced and knows  $\alpha$ , while fisherman 2 is inexperienced and doesn't know  $\alpha$ . Furthermore, assume  $E(\alpha) = \mu$  and  $\text{Var}(\alpha) = E(\alpha^2) - \mu^2 = \sigma^2$  completely characterize fisherman 2's beliefs about  $\alpha$  and that these beliefs are common information.

- a) Find the Bayesian Nash equilibrium efforts for each fisherman. Assume that the parameters yield an interior solution and that  $b > 0$ .

- b) How does total effort in this incomplete information game compare to the socially optimal total effort assuming there is no incomplete information? Assuming fisherman 2's beliefs about  $\alpha$  are unbiased (i.e.  $\alpha = \mu$ ), does fisherman 2's incomplete information move us closer to the optimal level of effort? That is, does fisherman 2's incomplete information exacerbate or mitigate the free riding incentives of these fishermen?

**ANSWER:**

- a) Fisherman 1's payoff is  $\pi_1 = \alpha^2 (x_1 + x_2) - 0.5\beta(x_1 + x_2)^2 - 0.5x_1^2$ . The first order condition is  $\pi_1' = \alpha^2 - \beta(x_1 + x_2) - x_1 = 0$  such that fisherman 1's best response is  $x_1(x_2, \alpha) = (\alpha^2 - \beta x_2) / (\beta + 1)$ . Fisherman 2's expected payoff is  $\pi_2 = E(\alpha^2 (x_1(\alpha) + x_2) - 0.5\beta(x_1(\alpha) + x_2)^2 - 0.5x_2^2)$ . The first order condition is  $\pi_2' = E(\alpha^2 - \beta(x_1(\alpha) + x_2) - x_2) = 0$  such that fisherman 2's best response is  $x_2(x_1(\alpha)) = E(\alpha^2 - \beta x_1(\alpha)) / (\beta + 1)$ . Plugging fisherman 1's best response into fisherman 2's and solving yields

$$x_2^* = \frac{m^2 + s^2}{2b + 1}. \text{ Plugging } x_2^* \text{ into fisherman 1's best response function yields}$$

$$x_1^*(a) = \frac{a^2}{(b + 1)} - \frac{b(m^2 + s^2)}{(b + 1)(2b + 1)}.$$

- b) For the socially optimal solution we want to maximize  $\pi_1 + \pi_2 = 2\alpha^2 (x_1 + x_2) - \beta(x_1 + x_2)^2 - 0.5x_1^2 - 0.5x_2^2$ . The first order conditions are  $2\alpha^2 - 2\beta(x_1 + x_2) - x_1 = 0$  and  $2\alpha^2 - 2\beta(x_1 + x_2) - x_2 = 0$ . Solving yields  $x_1^0 = x_2^0 = \frac{2a^2}{(4b + 1)}$  or  $x_1^0 + x_2^0 =$

$$\frac{4a^2}{(4b + 1)}. \text{ Note that total effort with incomplete information is } x_1^*(\alpha) + x_2^* =$$

$$\frac{(2b + 1)a^2 + (m^2 + s^2)}{(2b + 1)(b + 1)} \text{ or } x_1^*(\alpha) + x_2^* = \frac{2a^2}{(2b + 1)} + \frac{s^2}{(2b + 1)(b + 1)} \text{ if } \alpha = \mu.$$

Comparing  $x_1^0 + x_2^0$  to  $x_1^*(\alpha) + x_2^*$ :  $x_1^0 + x_2^0 > (<) x_1^*(\alpha) + x_2^* \Rightarrow$

$$\frac{4a^2}{(4b + 1)} > (<) \frac{2a^2}{(2b + 1)} + \frac{s^2}{(2b + 1)(b + 1)} \Rightarrow \sqrt{\frac{2(b + 1)}{(4b + 1)}} > (<) \frac{s}{a}.$$

Now if there was no incomplete information  $\sigma^2 = \sigma = 0$  implying  $x_1^0 + x_2^0 > x_1^*(\alpha) + x_2^*$ . Fishermen expend less than the socially optimal effort due to free riding incentives. Notice that  $x_1^*(\alpha) + x_2^*$  is increasing in  $\sigma^2$ , which means fisherman 2's incomplete information leads to an increase in equilibrium effort, which will move us closer to the optimum, mitigating free riding incentives. However, it is possible that fisherman 2's incomplete information could lead to too much fishing effort. In particular, if fisherman 2's beliefs are such that the coefficient of variation for  $\alpha$  ( $\sigma/\alpha$ ) is larger

$$\text{than } \sqrt{\frac{2(b + 1)}{(4b + 1)}}.$$

3. Suppose there are two firms competing for a government contract. Each knows its own value of the contract, but not the value to the other firm. Each also knows that they will win the contract only if they give the most generous bribe to G.W. Hedge. Assume firms are risk neutral. Also assume each firm's value is randomly drawn from the distribution function  $F(v)$  where  $v \in [0, V]$ .
- Find the symmetric and monotonic Bayesian Nash bribe strategy. Interpret the meaning of this strategy.
  - Assuming  $F(v)$  is uniform. Find the ex ante expected level of bribes for a firm. Compare this to the ex ante expected value of the contract. What is ex ante rent dissipation?

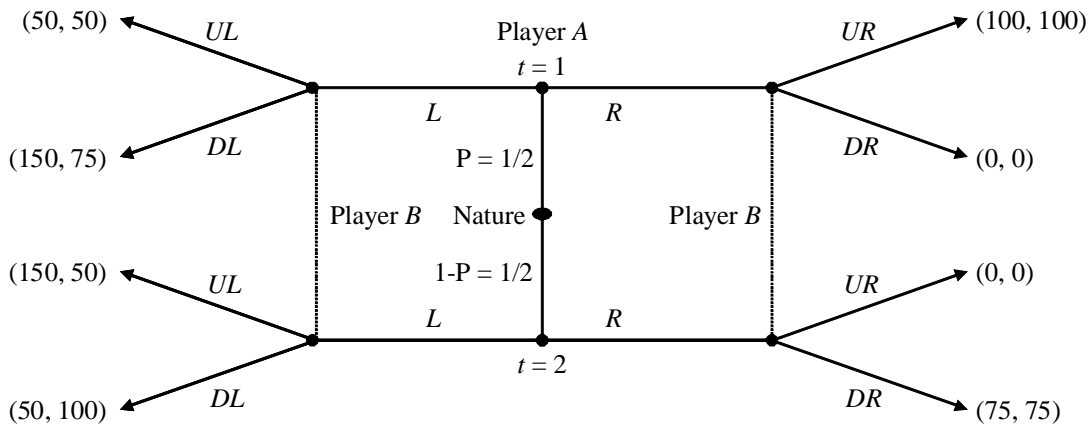
**ANSWER:**

- a) Firm  $i$ 's expected payoff is  $p_i = v_i \Pr(b_i > b_j(v_j)) - b_i$ . Assume  $b_i(v) = b_j(v) = b(v)$  such that  $b(v)$  is differentiable and monotonic. Then  $p_i = v_i F(b^{-1}(b_i)) - b_i$ . The first order condition is  $\frac{\partial p_i}{\partial x_i} = v_i f(x^{-1}(x_i)) \frac{\partial x^{-1}(x_i)}{\partial x_i} - 1 = 0$ . We will assume things are ok for the sufficiency conditions. With  $v_i = b^{-1}(b_i)$  and  $b_i = b(v_i)$ ,  $\frac{dv_i}{db_i} = (b^{-1})'(b_i)$  and  $\frac{db_i}{dv_i} = b'(v_i) = b'(b^{-1}(b_i))$ , so  $\frac{dv_i}{db_i} = (b^{-1})'(b_i)$   $\frac{dv_i}{db_i} = (b^{-1})'(b_i) = \frac{1}{b'(b^{-1}(b_i))} = \frac{1}{b'(v_i)}$ . The first order condition then implies  $b'(v_i) = v_i f(v_i)$ . Integrating up to  $v$  over  $v_i$ , then yields  $\int_0^v b'(v_i) dv_i = \int_0^v v_i f(v_i) dv_i$  or  $b(v) - b(0) = \int_0^v v_i f(v_i) dv_i$ . Now if  $v = 0$ , bribing anything greater than 0 yields  $p_i < 0$ , while bidding 0 would yield  $p_i = 0$ . Therefore,  $b(0) > 0$ , cannot be a best response, so  $b(v) = \int_0^v v_i f(v_i) dv_i$ . But what does this mean?
- Note that  $\int_0^v v_i f(v_i) dv_i = \int_0^v v_i \frac{f(v_i)}{F(v)} dv_i F(v) = E(v | v_i \leq v) \Pr(v_i \leq v)$ . So, a firm should bid their expected value conditional on their opponent's value being less times the probability that their opponent's value is less.

- b) For the uniform distribution,  $b(v) = \int_0^v \frac{v_i}{V} dv_i = \frac{v^2}{2V} - \frac{0^2}{2V} = \frac{v^2}{2V}$ . The ex ante expected value of a player's bribe is  $E(b(v)) = \int_0^V \frac{v^2}{2V^2} dv = \frac{V^3}{6V^2} = \frac{V}{6}$ . The ex ante expected value the contract is  $E(V) = \int_0^V \frac{v}{V} dv = \frac{V^2}{2V} = \frac{V}{2}$ , such that  $E(b(v)) / E(V) = 1/3$ . On

average, a firm should expend  $1/3$  of the value of the contract on bribes. Multiplying  $E(b(v))$  by 2 then yields the ex ante expected level of rent dissipation:  $E(R) = V/3$ .

4. Consider the dynamic incomplete information game in the figure below. There are three players: Nature, A, and B. Nature starts the game by randomly selecting A's type  $t$ , either 1 or 2, with equal probability. Nature reveals this information to A, but not to B. Once A has this information, it must choose either L or R. If A chooses L, B must choose between UL or DL. If A chooses R, B must choose between UR or DR. Payoffs are summarized in parentheses, with the first payoff corresponding to A and the second to B.
- What are A's and B's pure strategy spaces?
  - Find all pure strategy perfect Bayesian equilibria (PBE) for this game?
  - Which of these equilibria satisfy the intuitive criteria? Justify your answer.



**ANSWER:**

a) Player A's strategy space is  $\{(L, L), (R, L), (L, R), (R, R)\}$  where (*type 1's strategy*, *type 2's strategy*). Player B's strategy space is  $\{(UL, UR), (DL, UR), (UL, DR), (DL, DR)\}$ .

b) First, let's characterize B's best response given A chooses L and A chooses R.

If A chooses L, B earns  $50m_L + 50(1 - m_L)$  from choosing UL and  $75m_L + 100(1 - m_L)$  from choosing DL implying B's best response is DL.

If A chooses R, B earns  $100m_R + 0(1 - m_R)$  from choosing UR and  $0m_R + 75(1 - m_R)$  from choosing DR implying B's best response is UR for  $m_R \geq 3/7$  and DR for  $m_R \leq 3/7$ .

Now let's consider A's four possible pure strategies.

Let's start with (L, L) which implies  $m_L = 1/2$  and  $1 \geq m_R \geq 0$ . For  $m_L = 1/2$ , B's best response is DL. For  $m_R \leq 3/7$ , B's best response is DR, which means L is not a type 2 A's best response. For  $m_R \geq 3/7$ , B's best response is UR, which means L is a type 1 and 2 A's best response. So, one pure strategy perfect Bayesian equilibrium is  $\{(L, L), (DL, UR), m_L = 1/2, m_R \geq 3/7\}$ .

Now consider (R, L) which implies  $m_L = 0$  and  $m_R = 1$ . For  $m_L = 0$ , B's best response is DL. For  $m_R = 1$ , B's best response is UR. For a type 1 A, choosing L would yield 150, while choosing R would yield 100, so a type 1 A would want to deviate and this cannot be a perfect Bayesian equilibrium.

(L, R) implies  $m_L = 1$  and  $m_R = 0$ . For  $m_L = 1$ , B's best response is DL. For  $m_R = 0$ , B's best response is DR. For a type 1 A, choosing L would then yield 150, while choosing R would yield 0, so a type 1 A would not want to deviate. For a type 2 A, choosing L would then yield 50, while choosing R would yield 75, so a type 2 A would not want to deviate. So, another pure strategy perfect Bayesian equilibrium is  $\{(L, R), (DL, DR), m_L = 1, m_R = 0\}$ .

Finally, (R, R) which implies  $1 \geq m_L \geq 0$  and  $m_R = 1/2$ . For  $1 \geq m_L \geq 0$ , B's best response is DL. For  $m_R = 1/2$ , B's best response is UR. For a type 1 A, choosing L would then yield 150, while choosing R would yield 100, so a type 1 A would want to deviate and this cannot be a perfect Bayesian equilibrium.

c) Notice that DL is a best response for player B responding to L regardless of A's type. Therefore, for a type 1 A, R is equilibrium dominated by L ( $150 > 100 > 0$ ). Neither strategy is equilibrium dominated for a type 2 A ( $75 > 50 > 0$ ). Therefore, if B sees R has been selected, it should assume a type 2 A chose R. If a type 2 A chose R, Bayes rule would imply  $m_R = 0 < 3/7$  ruling out  $\{(L, L), (DL, UR), m_L = 1/2, m_R \geq 3/7\}$  and leaving the intuitive equilibrium  $\{(L, R), (DL, DR), m_L = 1, m_R = 0\}$ .

5. Suppose we have two auto company executives going to Capital Hill to get bailout money. Each executive needs a certain minimum amount of bailout funds in order to avoid bankruptcy, but will take as much as it can get—the more the better. How much is needed to avoid bankruptcy is not known to their competitor. Congress has a fixed amount of money it will hand out that is known to both executives. The executives also know that Congress does not want to handout money to a company if the amount it is asking for is not enough for the company to avoid bankruptcy. Finally, the executives know that Congress is mad and will give more of the bailout money to the company that appears to be less greedy. Construct a static game of incomplete information that captures the strategic incentives facing the executives. Derive Bayesian Nash equilibrium strategies for how much bailout money the executives will ask for given the minimum amount of funds they need to need to avoid bankruptcy.

**ANSWER:** You tell me!