Applied Game Theory
APEC 8205

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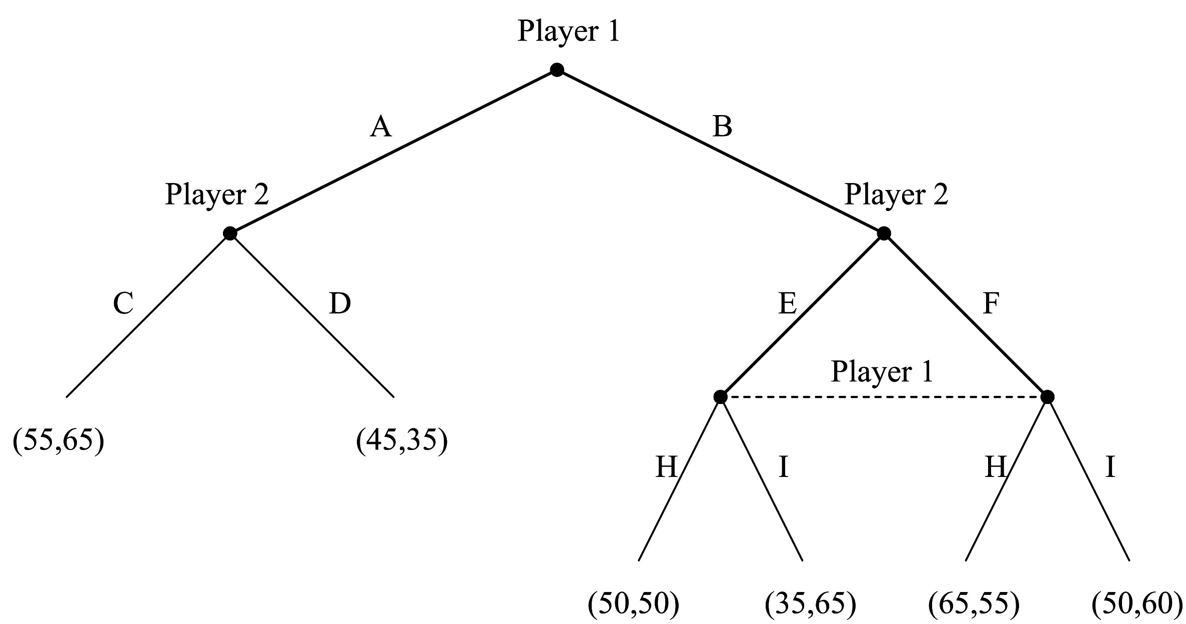
Answers to the Midterm Exam

1. Consider the extensive form game in Figure 1. The first payoff in parentheses is player 1's, while the second is player 2's. Assume a player's utility equals its payoff. (Each part worth 5 points)
   a. Write down the strategies for each player.
   b. Write down the normal form version of this game.
   c. What are all the pure strategy Nash equilibria?
   d. What is the pure strategy subgame perfect Nash equilibrium?
   e. Now suppose player i's utility is defined by $U_i = \pi_i - 2\times|\pi_i - \pi_j|$ where π_i and π_j are player i's and j's payoffs. What is the pure strategy subgame perfect Nash equilibrium now?
   f. Given the experimental evidence which of these equilibria seems most compelling? Explain.

Figure 1



Answer:

a. S_1 = {(A,H), (A,I), (B,H), (B,I)} and S_2 = {(C,E), (D,E), (C,F), (D,F)}.

b.

Player 2

Player 1	(C,E)	(D,E)	(C,F)	(D,F)
(A,H)	55*, 65*	45, 35	55, 65*	45, 35
(A,I)	55*, 65*	45, 35	55, 65*	45, 35
(B,H)	50, 50	50*, 50	65*, 55*	65*, 55*
(B,I)	35, 65*	35, 65*	50, 60	50, 60

c. Pure Strategy Nash Equilibria are {(A,H), (C,E)}; {(A,I), (C,E)}; {(B,H), (C,F)}; and {(B,H), (D,F)}.

d. There are three subgames: (i) the game as a whole, (ii) the game starting after player 1 chooses A, and (iii) the game starting after player 1 chooses B. Let us begin with (iii). The normal form is

Player 2

Player 1	E	F
H	50*, 50	65*, 55*
I	35, 65*	50, 60

which implies the subgame perfect Nash should include the strategy combination H and F for subgame (iii) with payoffs (65, 55).

For (ii), there is a pure strategy Nash equilibrium C with the payoffs (55, 65).

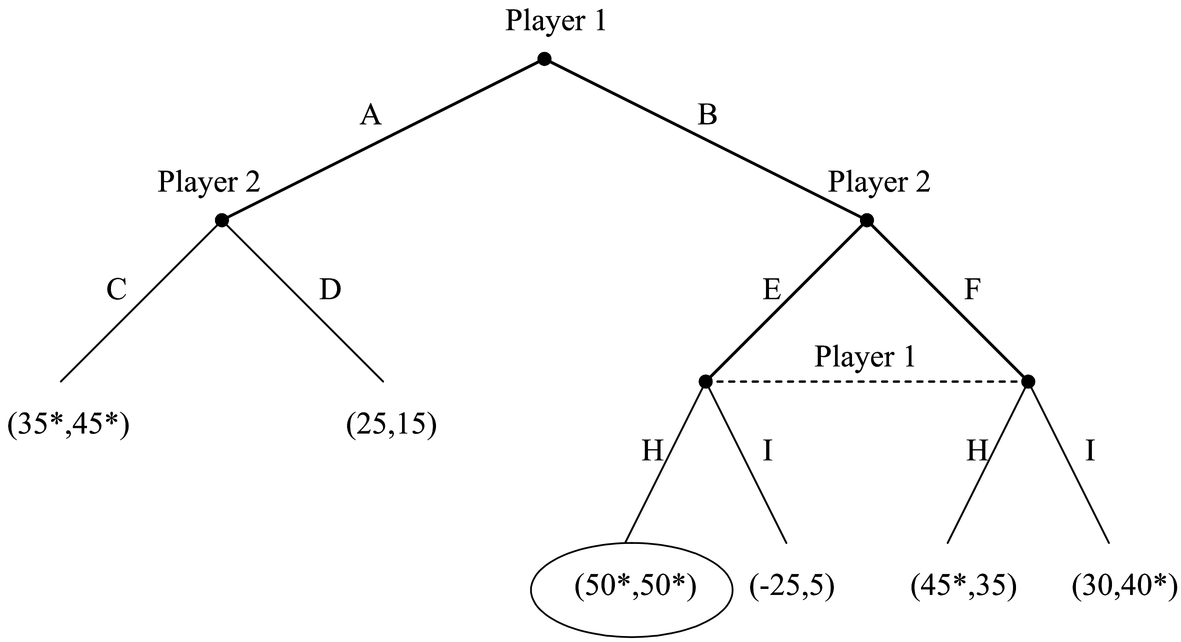
Comparing these two sets of payoffs, player 1 should choose B to start the game, so the subgame perfect equilibrium is {(B,H), (C,F)}.

e. Transforming the payoffs in Figure 1 yields Figure 1a below.

For this new game, the Nash equilibrium for (iii) is (H,E). The Nash equilibrium for (ii) is still C, so the subgame perfect equilibrium is {(B,H), (C,E)}.

f. Arguments can be made either way. The argument in favor of the equilibrium in part (d) is that these strategies yield both players high payoffs and each player acts in their own best self interest (classic game theory). Several arguments could be put forth the equilibrium in part (e) is the more compelling. The players care about equality of results they may prefer payoffs of (50, 50) to payoffs of (65, 55) even though both players get more in the latter case. It could also be argued that player 2 might resent the fact that player 1 did not choose A, which denies player 2 the chance to get a payoff of 65. Player 2 might reciprocate and play E (rather than F), which then denies player 1 the chance to get a payoff of 65. There is a lot of experimental evidence to suggest people do care about fair outcomes both in the sense of Rabin (1993) and in the sense of equal payoffs between players.

Figure 1a


Player 1
A
B
Player 2
Player 2
C
D
E
F
Player 1
(35*,45*)
(25,15)
H
I
H
I
(50*,50*)
(-25,5)
(45*,35)
(30,40*)

2. Suppose two defense contractors, L for Lockherd-Marin and B for Bowing, have the opportunity to compete for a portion of a government contract to produce the next generation of cargo plane. However, the government still has not determined how many planes it will need. Therefore, both Lockherd-Marin and Bowing have the opportunity to simultaneously invest lobbying dollars e_L and e_B in order to convince the government that it needs more planes. Specifically, the value of the government contract is $V = \alpha E - \beta E^2$ where $E = e_L + e_B$, $\alpha > 4$, and $\beta > 0$. Once the value of the contract is determined, Lockherd-Marin and Bowing must determine how the rewards of the contract are divided.
   a. (10 points) Suppose the government decides to let Lockherd-Marin choose its preferred division of the contract spoils (value) since it has more experience with these types of contracts. However, Bowing must approve the split or the government will give the contract to a third contractor, Hallibutton (e.g. an ultimatum game). What is each contractor's subgame perfect equilibrium lobbying effort and Lockherd-Marin's subgame perfect equilibrium offer to Bowing?
   b. (10 points) Suppose the proportion of the value of the contract awarded to contractor i is $P(x_i, x_j) = x_i/(x_i + x_j)$ for i = L, B where x_i and x_j are the ith and jth contractor's rent seeking expenditures once they know the value of the contract. Find the subgame perfect equilibrium lobbying and rent seeking expenditures for each contractor assuming they choose their rent seeking expenditures simultaneously.
   c. (5 points) Are total lobbying expenditures, E, for Lockherd-Marin and Bowing greater under the ultimatum game or the rent seeking game? Are Lockherd-Marin and Bowing's aggregate payoffs higher under the ultimatum game or the rent seeking game? Explain why your answer makes sense?

Answer:

a. For the subgame perfect equilibrium, we must first ask which types of offers Bowing should accept and which types it should reject. Note that this is just the ultimatum bargaining game, so Bowing should accept any positive offer, $\varepsilon > 0$.

   Now we must consider how much Lockherd-Marin should offer:

$$\max_{\varepsilon > 0} V - \varepsilon .$$

   The solution is the smallest possible ε regardless of V. If it makes this offer, Bowing will accept and the payoffs will be $V - \varepsilon$ for Lockherd-Marin and ε for Bowing.

   Now we can ask how much each contractor should devote to influencing the size of the contract. Bowing must solve:

$$\max_{e_B \geq 0} \varepsilon - e_B .$$

   Bowing best response function is simply $e_B(e_L) = 0$. Lockherd-Marin must solve:

$$\max_{e_L \geq 0} \ \alpha(e_L + e_B) - \beta(e_L + e_B)^2 - \varepsilon - e_L .$$

The first order condition for an interior solution is $\alpha - 2\beta(e_L + e_B) - 1 = 0$, implying the best response $e_L(e_B) = \frac{\alpha - 1}{2\beta} - e_B$. Solving for the best response functions, we get $e_L{*} = \frac{\alpha - 1}{2\beta}$, e_B*= 0, and $V{*} = \frac{\alpha^2 - 1}{4\beta}$. Bowing doesn't waste money lobbying for a larger contract because it knows Lockherd-Marin will only throw it a small scrap regardless of how big the contract is. $\pi_L{*} + \pi_B{*} = \frac{(\alpha - 1)^2}{4\beta}$

b. Here we need to start with the rent seeking game where the *i*th contractor must solve

$$\max_{x_i \geq 0} \ \frac{x_i}{x_i + x_j} V - x_i .$$

For an interior solution, this leads to the first order condition and best response function $\frac{x_j}{(x_i + x_j)^2} V - 1 = 0$ and $x_i(x_j) = \sqrt{x_j V} - x_j$. Solving our best response functions we get $x_L{*} = x_B{*} = \frac{V}{4}$, which implies equilibrium payoffs $\pi_L{*} = \pi_B{*} = \frac{V}{4}$.

Now we can ask how much each contractor should invest in lobbying effort:

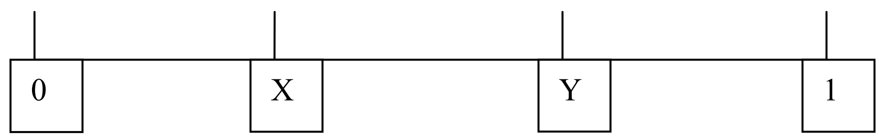
$$\max_{e_i \geq 0} \ \frac{\alpha(e_i + e_j) - \beta(e_i + e_j)^2}{4} - e_i .$$

The first order condition for an interior solution and the best response function are $\frac{\alpha - 2\beta(e_i + e_j)}{4} - 1 = 0$ and $e_i(e_j) = \frac{\alpha - 4}{2\beta} - e_j$. Solving the best response functions yields the subgame perfect Nash: $e_L{*} = e_B{*} = \frac{\alpha - 4}{4\beta}$, such that $V{*} = \frac{\alpha^2 - 16}{4\beta}$ and $x_L{*} = x_B{*} = \frac{\alpha^2 - 16}{16\beta}$.

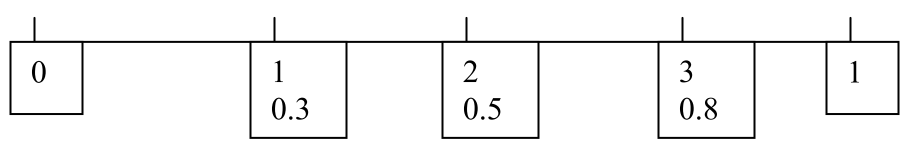
c. The equilibrium value of the contract is higher as is the aggregate payoff for the ultimatum game because $\frac{\alpha - 1}{2\beta} > \frac{\alpha - 4}{2\beta}$ when 3 > 0 and $\frac{(\alpha - 1)^2}{4\beta} > \frac{\alpha^2 - 16}{8\beta}$ when $(\alpha - 2) + 14 > 0$, which of course is always. More is invested in lobbying for the value of the contract and aggregate payoffs are higher because only one firm ultimately gets all the spoils of the lobbying effort and that is the only firm that has an incentive to lobby. There is no free riding on the other contractor's lobbying efforts. It is also worth noting that the division of the value of the contract is done without transactions costs in the ultimatum game. In the rent seeking game, the value of the contract is diminished by each contractor's rent-seeking activities.

3. There is to be an election between candidates. Suppose there is a continuum of voters uniformly distributed on a line from 0 to 1 where location represents where the voter is on the political spectrum from liberal to conservative. Voters prefer to vote for the candidate closest to them on the political spectrum. Because voters are uniformly distributed on a line of unit length, the length of the line segment of voters voting for a candidate equals that candidate's percentage of votes. For example, if voters to the left of α ($0 < \alpha < 1$) vote for candidate 1 and to the right of α vote for candidate 2 then candidate 1 gets α % of the vote and candidate 2 gets (1- α)%.

   a. (8 points) Suppose there are two candidates in the election. Candidate 1 is located at X and candidate 2 is located at Y, with $X < Y$ (see the figure below). Given X and Y, characterize which voters will vote for which candidate. Under what conditions for X and Y would candidate 1 win the election (by getting more than 50% of the vote)?



   b. (8 points) Suppose the two candidates simultaneously choose their locations (i.e., candidate 1 chooses X at the same time that candidate 2 chooses Y). What is the Nash equilibrium choice of X and Y? Note: if $X = Y$, candidates split the vote 50%-50%.
   c. (9 points) Now suppose there are three candidates in the election. Unlike part (b), suppose the candidates' locations are fixed, with candidate 1 located at 0.3, candidate 2 located at 0.5, and candidate 3 located at 0.8. Assume that voters form coalitions that vote as a block. Suppose there are four coalitions: a) those voters that prefer candidate 1, b) those voters that prefer candidate 2 and next prefer candidate 1, c) those voters that prefer candidate 2 and next prefer candidate 3, and d) those voters that prefer candidate 3. The election takes place in two stages. In the first stage, the candidate receiving the least votes is eliminated. The other two candidates then run against each other in the second stage. The candidate that receives more than 50% of the vote in the second stage wins the election. What is the proportion of voters in each coalition? What is the subgame perfect equilibrium strategy of each coalition? Which candidate wins in the subgame perfect equilibrium?



Answer:

a. Voters closer to X than to Y will prefer to vote for candidate 1 while those closer to Y than X will prefer to vote for candidate 2. All voters to the left of $\frac{X+Y}{2}$ are closer to X than to Y. Therefore, voters on the segment $\left[0, \frac{X+Y}{2}\right]$ will vote for candidate 1 while those on the segment $\left[\frac{X+Y}{2}, 1\right]$ will vote for candidate 2. Candidate 1 will win the election when $\frac{X+Y}{2} > \frac{1}{2}$; $X + Y > 1$. [Note: if $X + Y = 1$, the election is a tie. Then the candidate with the better lawyers will win (see Bush v. Gore (2000))].

b. Candidate 1 gets $\frac{X+Y}{2}$ percent of the vote when $X < Y$. This percentage is increasing in X. So candidate 1 should move to the right whenever she is to the left of candidate 2. On the other hand, if $X > Y$, candidate 1 gets $1 - \frac{X+Y}{2}$ percent of the vote, which is declining in X. So candidate 1 should move to the left whenever she is to the right of candidate 2. Therefore, candidate 1 would like to be as close to the position of candidate 2 as possible (but not equal). If $Y < ½$, the best response of candidate 1 is to be positioned just to the right of candidate 2: $X = Y + \varepsilon$. If $Y > ½$, the best response of candidate 1 is to be positioned just to the left of candidate 2: $X = Y - \varepsilon$. However, candidate 2 faces the exactly same logic yielding exactly the same type of best response function. Therefore, there is no pair of best responses for either $X \neq ½$ or $Y \neq ½$. If the rival candidate is positioned at ½, the best response is also to choose a position of ½. By doing so, each candidate gets 50% of the vote. Any choice other than ½ gives that candidate less than 50% of the vote. Therefore, the (unique) Nash equilibrium is for candidate 1 to set $X = ½$ and for candidate 2 to set $Y = ½$.

c. Everyone to the left of 0.4 prefers candidate 1 so that coalition 1 has 40% of the voters. Everyone between 0.4 and 0.65 prefers candidate 2. However, some of those who prefer candidate 2 prefer candidate 1 to candidate 3, while others prefer candidate 3 to candidate 1. Candidate 1 is preferred to candidate 3 for everyone to the left of 0.55. Coalition 2 consists of voters in the range from [0.4, 0.55], which means it has 15% of voters. Coalition 3 consists of voters from [0.55, 0.65], which means it has 10% of voters. Everyone to the right of 0.65 votes for candidate 3, which means that coalition 4 has 35% of voters.

   In the second stage, there are three possible runoff elections: i) 1 vs. 2; ii) 1 vs 3; and iii) 2 vs 3.

   - Coalition 1: would vote for candidate 1 in case (i) and (ii), and would vote for candidate 2 in case (iii).
   - Coalition 2: would vote for candidate 1 in case (ii), and would vote for candidate 2 in cases (i) and (iii).

- Coalition 3: would vote for candidate 3 in case (ii), and would vote for candidate 2 in cases (i) and (iii).
- Coalition 4: would vote for candidate 2 in case (i), and would vote for candidate 3 in cases (ii) and (iii).

Combining these together, we find that in the second stage:

- Candidate 2 wins versus candidate 1 (case (i)): 60% to 40%
- Candidate 1 wins versus candidate 3 (case (ii)): 55% to 45%
- Candidate 2 wins versus candidate 3 (case (iii)): 65% to 35%

In the first stage, strategic voters would realize that the campaign is really between candidate 1 and candidate 2 because there is no possibility that candidate 3 can win in the second stage. Coalition 1 prefers candidate 1; they should vote for candidate 1. Coalitions 2 and 3 prefer candidate 2; they should vote for candidate 2. Coalition 4, however, faces a problem. While they like candidate 3 the best, there is no chance that candidate 3 will win. If they vote for candidate 3 in the first stage, the second stage will feature candidate 1 versus candidate 3 and candidate 1 will win. They would rather have candidate 2 than candidate 1. Therefore, coalition 4 should abandon candidate 3 in the first stage and vote for candidate 2. By doing so, the top two vote getters in stage 1 are candidates 1 and 2. Candidate 2 will get more than 50% of the votes in the second stage and will win the election.