Applied Game Theory
APEC 8205

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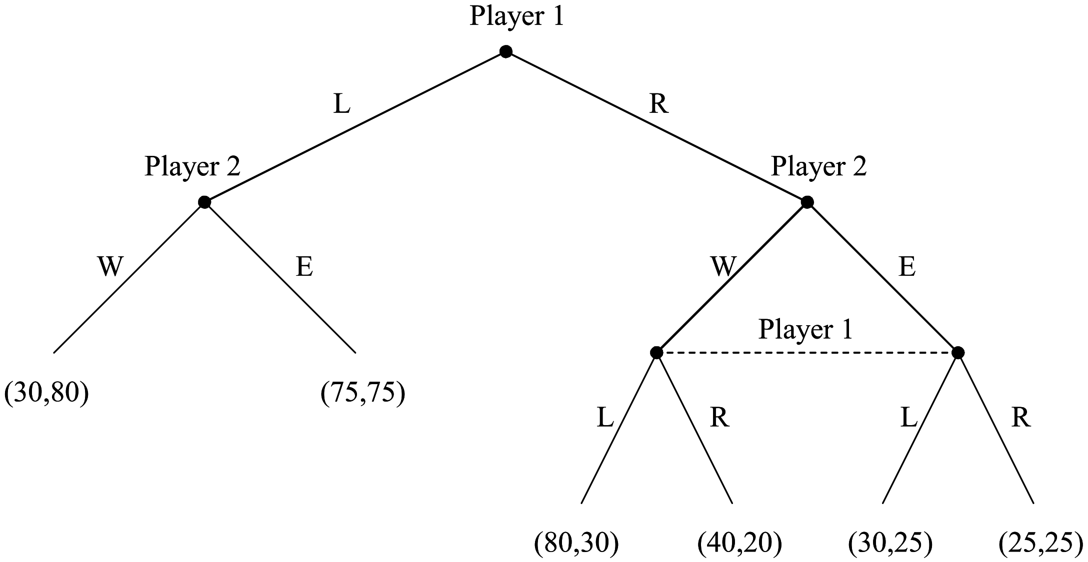
Midterm Exam

1. Consider the extensive form game in Figure 1. The first payoff in parentheses is player 1's, while the second is player 2's. Assume a player's utility equals its payoff.
   a. Write down the strategies for each player.
   b. Write down the normal form version of this game.
   c. What are the pure strategy Nash equilibria?
   d. What is the pure strategy subgame perfect Nash equilibrium?
   e. Now suppose player i's utility is defined by $U_i = \pi_i - 0.5|\pi_i - \pi_j|$ where π_i and π_j are player i's and j's payoffs. What is the pure strategy subgame perfect Nash equilibrium now?
   f. Given the experimental evidence which of these equilibria seems most compelling? Explain.

Figure 1



Answer:

a. Player 1's strategies are $S_1 \in$ {(L,L), (L,R), (R,L), (R,R)} where (strategy choice to start the game, strategy choice if game starts with R). Player 2's strategies are $S_2 \in$ {(W,W), (W,E), (E,W), (E,E)} where (strategy choice if player 1 chooses L, strategy choice if player 1 choose R).

b.

		Player 2			
		(W,W)	(W,E)	(E,W)	(E,E)
	(L,L)	80* 30	80* 30*	75 75	75 75*
Player 1	(L,R)	80* 30	80* 30*	75 75	75 75*
	(R,L)	30* 80*	25 30*	30* 80*	25 30
	(R,R)	20 40	25* 25	20 40	25* 25

c. The pure strategy Nash equilibria include {(R,L),(W,W)}, {(L,L),(W,E)}, {(L,R),(W,E)}, and {(R,L),(E,W)}.

d. There are three subgames: (i) the game as a whole, (ii) the game starting after player 1 chooses L, and (iii) the game starting after player 1 chooses R. Let us begin with (iii). The normal form is

		Player 2	
		W	E
Player 1	L	30* 80*	25 30*
	R	20 40	25* 25

which has the pure strategy Nash equilibria {L,W} that yield payoffs (80,30).

For (ii), there is a pure strategy Nash equilibrium W with the payoffs (30,80).

Now for the game as a whole, (80,30) is better for Player 1 than (30,80), so Player 1 should start the game with R, which leads to the unique subgame perfect equilibrium {(R,L),(W,W)}.

e. Transforming each players payoffs leads to the game in Figure 1' below.

For subgame (iii), there is a unique Nash equilibrium (L,E) with payoffs (27.5, 22.5).

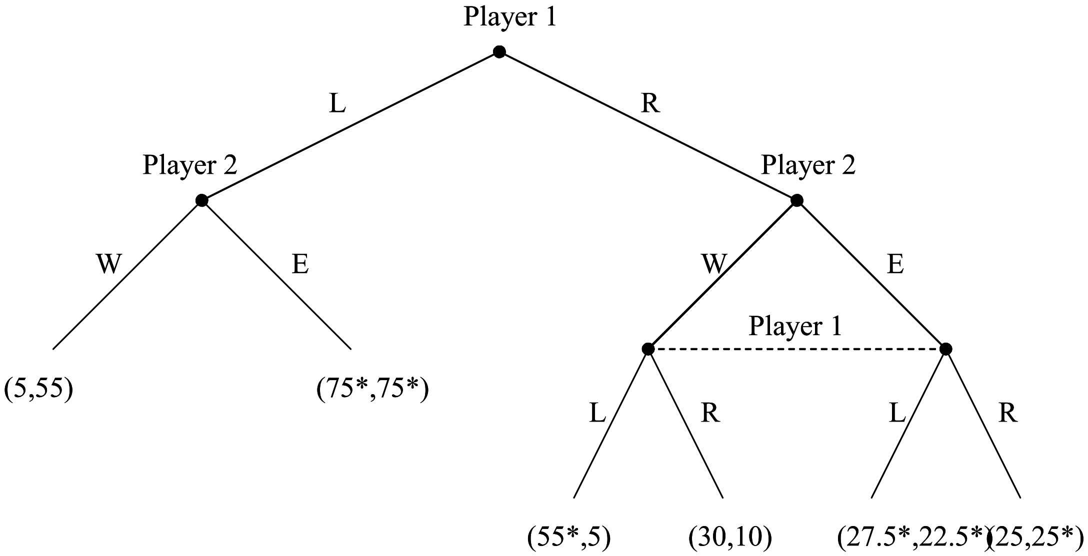
For subgame (ii), there is a unique Nahs equilibrium E with payoff (75,75).

For the game as a whole, Player 1 would prefer 75 to 27.5, so the unique subgame perfect equilibrium is {(L,L),(E,E)}.

f. I would argue that the equilibrium in part e is the most compelling because if Player 1 chooses R followed by Player 2 choosing E both players payoffs are 75, while all the Nash equilibria for the game have one player's payoff equal to 80 and the other's equal to 30. Therefore, none of the Nash equilibrium are particularly fair given the possibility that one player could earn substantially more and the other only a little less. The utility function used in the solution in part e makes the notion of fairness explicit and results it what I would say is a more sensible equilibrium.

The game is like a trust game, but there are also overtones of spite. Player 1 could choose L trusting Player 2 to do the right thing by choosing E. If he doesn't trust Player 2 to respond to L with E, then he should choose R followed by L, which would guarantee him at least as much as choosing L to start. If Player 1 chooses R to start the game, Player 2 should choose W, but might choose E to spite Player 1. Can you think of a utility function with spite in it that would lead to the same conclusion?

Figure 1'



2. Suppose there are four firms that produce an identical good. Define q_i as the quantity produced by firm *i*, $i = 1, 2, 3, 4$. The inverse market demand curve for this good is $P(Q) = a - Q$, where *P* is the price of the good and $Q = \sum_{i=1}^{4} q_i$. Each firm can produce this good at a constant per unit cost of $c > 0$. Assume that $a > c$.
   a. Solve for the (Cournot) Nash equilibrium when all firms simultaneously choose quantity. Show that profit for firm *i*, Π_i, is equal to the square of equilibrium quantity choice, $(q_i*)^2$.
   b. Now suppose that firms collude and maximize industry profits. Assuming that each firm produces ¼ of the collusive output, how much does each firm produce and how much profit does it make?
   c. What is the profit maximizing quantity choice for firm *i* given all other firms abide by the collusive scheme? How much profit does firm *i* make?
   d. Now suppose that these firms play an infinitely repeated game in which they simultaneously choose quantity in each round. Let $0 < \delta < 1$ be the discount factor applied to payoffs between each round. For what values of δ can the collusive outcome in each round be supported as a subgame perfect equilibrium in the repeated game?
   e. Finally, suppose that the market is growing over time. Suppose that $(a\text{-}c)$ grows by $\gamma > 1$ between each round. For what values of δ can the collusive outcome in each round be supported as a subgame perfect equilibrium in the repeated game? Is the collusive outcome supportable as a subgame perfect equilibrium for a wider or narrower range of δ than the answer for part (d)?

Answer:

a. Each firm faces the following profit maximizing problem:

$$Max\ (a - Q)q_i - cq_i$$

Taking the first order condition and solving for Nash equilibrium we find:

$$q_i* = \frac{a-c}{5}$$

Plugging in the equilibrium value for strategies into profit function:

$$\Pi_i = [a - c - \frac{4(a-c)}{5}][\frac{(a-c)}{5}]$$

$$= \frac{(a-c)^2}{25} = (q_i*)^2$$

b. The collusive output (monopoly solution):

$$Max\ (a - Q)Q - cQ$$

$$Q* = \frac{(a-c)}{2}$$

Each firm will produce:

$q_i = \frac{(a-c)}{8}$ and earn profit of $\Pi_i = \frac{(a-c)^2}{16}$.

c. Firm *i*'s profit maximization problem in this case is

$$Max\ (a - \frac{3(a-c)}{8} - q_i)q_i - cq_i$$

$$= Max\ (\frac{5(a-c)}{8} - q_i)q_i$$

The optimal choice for firm *i* is:

$$q_i = \frac{5(a-c)}{16}.$$

Firm i will earn profit of $\Pi_i = \left(\frac{5(a-c)}{16}\right)^2 = \frac{25(a-c)^2}{256}$.

d. The collusive outcome in each round can be supported as a subgame perfect equilibrium in the repeated game if and only if:

$$\frac{(a-c)^2}{16(1-\delta)} \geq \frac{25(a-c)^2}{256} + \frac{\delta(a-c)^2}{25(1-\delta)}$$

$$\delta \geq \frac{225}{369} = \frac{25}{41} = 0.61.$$

e. The collusive outcome in each round can be supported as a subgame perfect equilibrium in the repeated game if and only if:

$$\frac{(a-c)^2}{16(1-\delta\gamma)} \geq \frac{25(a-c)^2}{256} + \frac{\delta\gamma(a-c)^2}{25(1-\delta\gamma)}$$

$$\delta \geq \frac{225}{369\gamma} = \frac{25}{41\gamma} = \frac{0.61}{\gamma} < 0.61.$$

In a growing market, even lower discount factors can support the collusive outcome, i.e., there is a wider range of discount factors that support collusion.

3. Consider two firms that produce a similar product and compete over price. The marginal cost of production is normalized to zero for both firms. Suppose the first firm has the opportunity to advertise its product before production occurs, while the second firm does not. By advertising its product firm 1 can command a price premium, such that $p_1 = p_2 + \delta$ where $\delta \geq 0$. Demand for firm 1's and 2's products are $q_1 = \begin{cases} 0, & for\ p_1 > p_2 + \delta \\ 0.5(10 + \delta - p_1), & for\ p_1 = p_2 + \delta \\ 10 + \delta - p_1, & for\ p_1 < p_2 + \delta \end{cases}$ and $q_2 = \begin{cases} 0, & for\ p_1 < p_2 + \delta \\ 0.5(10 - p_2), & for\ p_1 = p_2 + \delta \\ 10 - p_2, & for\ p_1 > p_2 + \delta \end{cases}$. Firm 1's cost of advertising in terms of the price premium is $c(\delta) = \delta^2$. The game facing the firms is a two stage game. In the first stage, firm 1 chooses how much to advertise or analogously, how high to set its price premium. In the second stage, firms choose price simultaneously and markets clear. Find the subgame perfect Nash equilibrium prices and price premium (i.e.p_1, p_2, and δ). Does firm 1's advertising help or hurt firm 2? Explain.

Answer:

To find the subgame perfect equilibrium we must work backward. Starting in the second stage of the game, each firm's profit as a function of its opponent's strategy are

$$\pi_1(p_2) = \begin{cases} -\delta^2, & for\ p_1 > p_2 + \delta \\ p_1 \dfrac{10 + \delta - p_1}{2} - \delta^2, & for\ p_1 = p_2 + \delta \\ p_1(10 + \delta - p_1) - \delta^2, & for\ p_1 < p_2 + \delta \end{cases}$$

and

$$\pi_2(p_1) = \begin{cases} p_2(10 - p_2), & for\ p_1 > p_2 + \delta \\ p_2 \dfrac{10 - p_2}{2}, & for\ p_1 = p_2 + \delta \\ 0, & for\ p_1 < p_2 + \delta \end{cases}$$

For $p_1 > 0$, choosing $p_1 > p_2 + \delta$ is a strictly dominated strategy for firm 1, so any strategy combinations with $p_1 > p_2 + \delta$ and $p_1 > 0$ cannot be part of a Nash equilibrium.

For $p_2 > 0$, choosing $p_1 > p_2 + \delta$ is a strictly dominated strategy for firm 2, so any strategy combinations with $p_1 > p_2 + \delta$ and $p_2 > 0$ cannot be part of a Nash equilibrium.

Therefore, for $p_1 > 0$ and $p_2 > 0$, potential Nash equilibria must be such that $p_1 = p_2 + \delta$. Consider $p_1' > 0$ and $p_2' > 0$ such that $p_1' = p_2' + \delta$. Suppose firm 1 raised its price from p_1' to $p_1'\varepsilon$ where $\varepsilon > 1$. Its profit would be strictly lower because

$p_1'\frac{10+\delta-p_1'}{2}-\delta^2$ > $-\delta^2$. A similar argument rules out the possibility of firm 2 doing better by raising its price. Now suppose firm 1 lowered its price from p_1' to p_1'ε where 1 > ε > 0. For p_1' to be a best response, $p_1'\frac{10+\delta-p_1'}{2}-\delta^2 \geq$ $p_1'\varepsilon(10+\delta-p_1'\varepsilon)-\delta^2$ or $(1-2\varepsilon)\geq\frac{p_1'}{10+\delta}(1-2\varepsilon^2)$. As ε approaches 1, 1 - 2ε and 1 - $2\varepsilon^2$ approach -1 or $(1-2\varepsilon)=-1<-\frac{p_1'}{10+\delta}=\frac{p_1'}{10+\delta}(1-2\varepsilon^2)$ for any p_1' resulting in $q_1 > 0$. Since 1 - 2ε and 1 - $2\varepsilon^2$ are continuous in ε, we can always find an ε such that $p_1'\frac{10+\delta-p_1'}{2}-\delta^2$ < $p_1'\varepsilon(10+\delta-p_1'\varepsilon)-\delta^2$, which implies p_1' is not a best response.

This leaves three remaining possibilities to consider: (i) $p_1 = 0$ and $p_2 > 0$, (ii) $p_1 = 0$ and $p_2 = 0$, and (iii) $p_1 > 0$ and $p_2 = 0$.

Case (i) is easily dismissed by the assumption $\delta \geq 0$, which implies the contradiction $p_1 \geq p_2$.

Case (ii) implies $p_1 < p_2 + \delta$ for $\delta > 0$. Neither player can decrease their price unilaterally, but each could raise their price unilaterally. For firm 2, any increase in price will still result in $p_1 \leq p_2 + \delta$ and 0 profit, so $p_2 = 0$ is a best response to $p_1 = 0$. For firm 1, a small increase in price will result in $p_1(10 + \delta - p_1) - \delta^2 > - \delta^2$, so firm 1 can do better by unilaterally deviating. Bottom line is that this is not a Nash equilibrium when it is optimal for firm 1 to invest in advertising.

This leaves Case (iii). We know from above that choosing $p_1 > p_2 + \delta = \delta$ is strictly dominated. If firm 1 chooses $p_1 = \delta$, $\pi_1 = 5\delta - \delta^2$ and firm 2 choosing $p_2 = 0$ is still a best response because increasing p_2 would not affect its profit. If firm 1 chooses $p_1 < \delta$, it should choose $p_1 = (10 + \delta)/2$ if $(10 + \delta)/2 < \delta$ or $p_1 = \delta - \varepsilon$ if $(10 + \delta)/2 \geq \delta$ for ε as close as possible to 0. For $p_1 = (10 + \delta)/2$, $\pi_1 = ((10 + \delta)/2)^2 - \delta^2$. for $p_1 = \delta - \varepsilon$, $\pi_1 = (\delta - \varepsilon)(10 - \varepsilon) - \delta^2$. In either case, firm 2 would still not have an incentive to deviate.

Note that $((10 + \delta)/2)^2 - \delta^2 > 5\delta - \delta^2$, so if $p_1 = (10 + \delta)/2 < \delta$, the Nash equilibrium prices are $p_1^* = (10 + \delta)/2$ and $p_2^* = 0$. Alternatively, if $(\delta - \varepsilon)(10 - \varepsilon) - \delta^2 > 5\delta - \delta^2$, the Nash equilibrium prices are $p_1^* = \delta - \varepsilon$ and $p_2^* = 0$. If $(\delta - \varepsilon)(10 - \varepsilon) - \delta^2 < 5\delta - \delta^2$, the Nash equilibrium prices are $p_1^* = 0$ and $p_2^* = 0$.

Now to the first stage of the game. First consider, $p_1^* = (10 + \delta)/2$ and $p_2^* = 0$. Firm 1's profit is $\pi_1 = ((10 + \delta)/2)^2 - \delta^2$. The first order condition is $(10 - 3\delta)/2 = 0$, so $\delta^* = 10/3$, but this implies $\delta < 10$, which contradicts $(10 + \delta)/2 < \delta$ required for the Nash equilibrium in stage 1.

Next consider, $p_1* = \delta - \varepsilon$ and $p_2* = 0$. Firm 1's profit is $\pi_1 = (\delta - \varepsilon)(10 - \varepsilon) - \delta^2$. The first order condition is $10 - \varepsilon - 2\delta = 0$, so $\delta^* = (10 - \varepsilon)/2$, which may or may not satisfy $(\delta - \varepsilon)(10 - \varepsilon) - \delta^2 > 5\delta - \delta^2$ depending on the magnitude of ε. If $(\delta - \varepsilon)(10 - \varepsilon) - \delta^2 > 5\delta - \delta^2$ is satisfied, the subgame perfect equilibrium will be $\delta^* = (10 - \varepsilon)/2$, $p_1* = (10 - 3\varepsilon)/2$, and $p_2* = 0$.

Finally, consider, $p_1* = 0$ and $p_2* = 0$. Firm 1's profit is $\pi_1 = 5\delta - \delta^2$. The first order condition is $5 - 2\delta = 0$, so $\delta^* = 5/2$, which satisfies $(10 + \delta)/2 < \delta$ and may or may not satisfy $(\delta - \varepsilon)(10 - \varepsilon) - \delta^2 < 5\delta - \delta^2$ depending on the magnitude of ε. If $(\delta - \varepsilon)(10 - \varepsilon) - \delta^2 < 5\delta - \delta^2$ is satisfied, the subgame perfect equilibrium will be $\delta^* = 5/2$, $p_1* = 0$, and $p_2* = 0$.

Given the solutions to these last two cases $(\delta - \varepsilon)(10 - \varepsilon) - \delta^2 >(<) 5\delta - \delta^2$ when $300 – 160\varepsilon + 12\varepsilon^2 >(<)0$. So $\delta^* = (10 - \varepsilon)/2$, $p_1* = (10 - 3\varepsilon)/2$, and $p_2* = 0$ when $300 – 160\varepsilon + 12\varepsilon^2 > 0$ and $\delta^* = 5/2$, $p_1* = 0$, and $p_2* = 0$ when $300 – 160\varepsilon + 12\varepsilon^2 < 0$.

In the standard, Bertrand model neither firm earn positive profit. So, firm 1's investment in advertising does not hurt or help firm 2 because firm 2 always earns 0 profit in equilibrium. Advertising does however benefit firm 1 because it can use advertising to secure a price in excess of its marginal cost of production, which earns it positive profit.