# Game Theory
# APEC 8205

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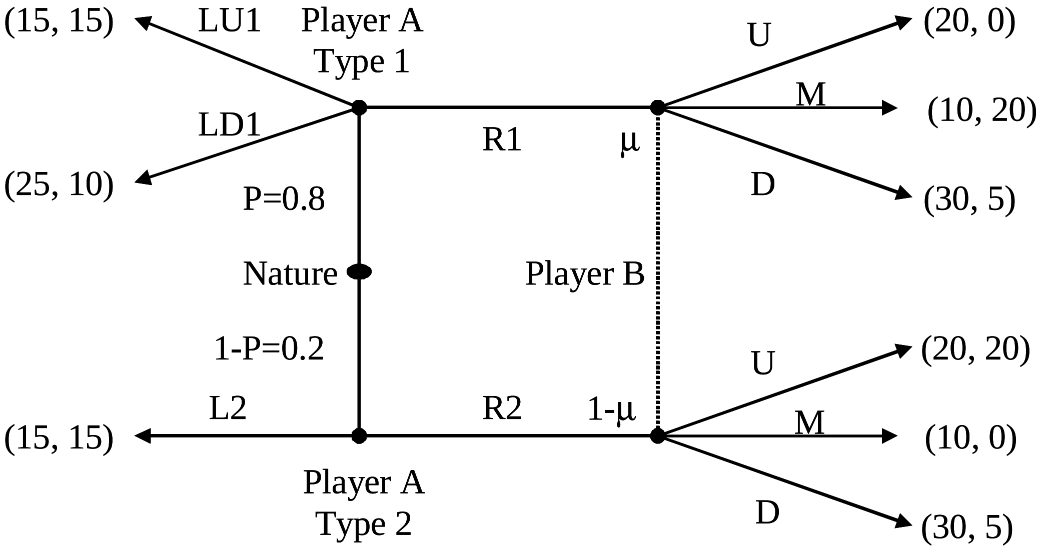
## Final Exam Answers

Please answer all of the following questions. Please be clear and concise in your answers (define terms and label figures). This exam is open mind but closed book. Time limit: 3 hours. Think before writing. Relax. Do well.

1. Consider the game in the Figure below. Nature starts by choosing player A's type (either 1 with probability P = 0.8 or 2 with probability 1 - P = 0.2). This information is then revealed to player A, but not player B. Player A then gets to move. Player A's strategy choices depend on its type. Type 1 can choose LU1 or LD1 to end the game immediately or R1 to let player B move. Type 2 can choose L2 to end the game immediately or R2 to let player B move. If player B gets to move, it gets to select U, M, or D without getting to see whether it was a type 1 A giving him that chance move or a type 2 A. Payoffs are summarized in parentheses with the first corresponding to player A and the second to player B. Let μ represent player B's belief that A is type 1 given it gets a chance to move.

   a) Write down the strategy space for each player.
   b) Write down the normal form for this game.
   c) Find all of the pure strategy Bayesian Nash equilibria.
   d) Find all of the pure strategy perfect Bayesian equilibria.
   e) Explain which of these equilibria seems more plausible based on the logic of the intuitive criterion.



(A's Payoff, B's Payoff)

**Answer:**

a) $S_A$ = {(LU1,L2), (LD1,L2), (R1,L2), (LU1,R2), (LD1,R2), (R1,R2)} for player A. $S_B$ = {U, M, D} for player B.

b)

| | | **Player B** | | |
|---|---|---|---|---|
| | | U | M | D |
| **Player A** | (LU1,L2) | (4×15 + 15)/5=15*<br>(15,15) | (4×15 + 15)/5=15*<br>(15,15*) | (4×15 + 15)/5=15*<br>(15,15) |
| | (LD1,L2) | (4×10 + 15)/5=11*<br>(25*,15) | (4×10 + 15)/5=11*<br>(25*,15*) | (4×10 + 15)/5=11*<br>(25,15) |
| | (R1,L2) | (4× 0+ 15)/5=3<br>(20,15) | (4×20 + 15)/5=19*<br>(10,15*) | (4×5 + 15)/5=7<br>(30*,15) |
| | (LU1,R2) | (4×15 + 20)/5=16*<br>(15,20*) | (4×15 + 0)/5=12<br>(15,10) | (4×15 + 5)/5=13<br>(15,30*) |
| | (LD1,R2) | (4×10 + 20)/5=12*<br>(25*,20*) | (4×10 + 0)/5=8<br>(25*,10) | (4×10 + 5)/5=9<br>(25,30*) |
| | (R1,R2) | (4× 0+ 20)/5=4<br>(20,20*) | (4× 20+ 0)/5=16*<br>(10,10) | (4×5 + 5)/5=5<br>(30*,30*) |

c) I have put * by each player best response in the normal form table above. The pure strategy Bayesian Nash equilibria are then {(LD1,R2),U} and {(LD1,L2),M}.

d) For a perfect Bayesian equilibrium, we need sequential rationality to hold with beliefs updated by Bayes rule where possible. We only need to consider the two Bayesian Nash equilibria because the perfect Bayesian equilibrium is a refinement of the Bayesian Nash equilibrium.

First consider {(LD1,R2),U}. Since only a type 2 player A will give player B the move in this equilibrium, Bayes rule implies $\mu = 0$. If $\mu = 0$, player B's best response is U. If player B chooses U, a type 1 player A's best response is LD1, while a type 2 player A's best response is U. Therefore, {(LD1,R2),U, $\mu = 0$} is a perfect Bayesian equilibrium.

Now consider {(LD1,L2),M}. Since no A player ever gives player B the move Bayes rule does not apply so $\mu \in [0,1]$ can potentially be an equilibrium belief. But sequential rationality says $\mu$ must be such that choosing (LD1,L2) is a best response for player A. This is only the case if player B's best response is to choose M. For M to be optimal for player B two conditions must hold: $20\mu + 0(1 - \mu) \geq 0\mu + 20(1 - \mu)$ and $20\mu + 0(1 - \mu) \geq 5\mu + 5(1 - \mu)$. For $20\mu + 0(1 - \mu) \geq 0\mu + 20(1 - \mu)$, $\mu \geq 1/2$. For $20\mu + 0(1 - \mu) \geq 5\mu + 5(1 - \mu)$, $\mu \geq 1/4$. Therefore, {(LD1,L2),M, $\mu \geq 1/2$} is a perfect Bayesian equilibrium.

e) If {U, M, D} are all possible strategies for player B, then there is no potential to apply equilibrium domination and rule out strategies via the Intuitive Criterion. However, D can never be a best response for player B. Playing D generates a payoff of 5. For $\mu \leq 1/2$, U is a best response and generates a payoff of greater than 5. For $\mu \geq 1/2$, M is a best response and generates a payoff of greater than 5. If D is not a best response for B, there is no potential for a player A of type 1 to deviate and be better off for any possible off the equilibrium path belief (25>20 for U and 25 > 10 for M). However, there is potential for a player A of type 2 to deviate and be better off if the off the equilibrium path belief is $\mu \leq ½$ because player B would choose U yielding a type 2 player A 20, which is greater than 15 from playing the equilibrium action L2. If it is only reasonable for a type 2 player A to deviate, Bayes rule implies player B's belief should be $\mu = 0$, which contradicts {(LD1,L2),M, $\mu \geq 1/2$}. Therefore, with the Intuitive Criterion we can rule out everything except {(LD1,R2),U, $\mu = 0$}.

2. Suppose we have two fishermen. Each expends effort $x_i$ for i = 1,2 to catch fish. The total catch, F, is a quadratic function of total effort: $F(X) = 2(\alpha^2 X - 0.5\beta X^2)$ where $X = x_1 + x_2$ and each fisherman shares in the catch equally. The price of fish is normalized to 1, while the cost of effort is $C(x_i) = 0.5x_i^2$. Finally, suppose fisherman 1 is experienced and knows $\alpha^2$, while fisherman 2 is inexperienced and doesn't know $\alpha^2$. Furthermore, assume $E(\alpha) = \mu$ and $Var(\alpha) = E(\alpha^2) - \mu^2 = \sigma^2$.

   a) Find the Bayesian Nash equilibrium efforts for each fisherman. *Assume that the parameters yield and interior solution and that b > 0.*
   b) How does fisherman 2's lack of experience affect the equilibrium efforts of both fishermen and total effort? That is, how does an increase in the variance of $\alpha$ affect individual and total effort? Does this result make sense? Explain.

**Answer:**

a) Fisherman 1's payoff is $\pi_1 = \alpha^2 (x_1 + x_2) - 0.5\beta(x_1 + x_2)^2 - 0.5x_1^2$. The first order condition is $\pi_1' = \alpha^2 - \beta(x_1 + x_2) - x_1 = 0$ such that fisherman 1's best response is $x_1(x_2, \alpha) = (\alpha^2 - \beta x_2)/ (\beta + 1)$. Fisherman 2's expected payoff is $\pi_2 = E(\alpha^2 (x_1(\alpha) + x_2) - 0.5\beta(x_1(\alpha) + x_2)^2 - 0.5x_2^2)$. The first order condition is $\pi_2' = E(\alpha^2 - \beta(x_1(\alpha) + x_2) - x_2) = 0$ such that fisherman 2's best response is $x_2(x_1(\alpha)) = E(\alpha^2 - \beta x_1(\alpha))/(\beta + 1)$. Plugging fisherman 1's best response into fisherman 2's and solving yields $x_2 * = \frac{m^2 + s^2}{2b + 1}$. Plugging $x_2*$ into fisherman 1's best response function yields $x_1 *(a) = \frac{a^2}{(b+1)} - \frac{b(m^2 + s^2)}{(b+1)(2b+1)}$.

b) Notice that as $\sigma^2$ approaches 0 $\mu^2$ approaches $\alpha^2$ and $x_1*(\alpha)$ approaches $x_2*$. Taking the derivation of $x_1*(\alpha)$ and $x_2*$ with respect to $\sigma^2$ yields $\frac{\partial x_1 *(a)}{\partial s^2} = \frac{-b}{(b+1)(2b+1)} < 0$ and $\frac{\partial x_2 *}{\partial s^2} = \frac{(b+1)}{(b+1)(2b+1)} > 0$ such that $\frac{\partial x_1 *(a)}{\partial s^2} + \frac{\partial x_2 *}{\partial s^2} = \frac{1}{(b+1)(2b+1)} > 0$. So, increased uncertainty decrease fisherman 1's effort, increases fisherman 2's effort, and increases overall effort. The intuition for this result is as follows. Because $\alpha^2$ is in the objective function (not simply $\alpha$), the objective function is convex with respect to $\alpha$. In this case, uncertainty about $\alpha$ increases the expected payoff (i.e., the objective function is convex is $\alpha$), so that fisherman 2 increases effort with uncertainty.  Fisherman 1 responds to the increased effort by decreasing effort but does not fully offset changes by Fisherman 2, resulting in an overall increase in fishing effort.

3. Two firms compete by choosing quantities simultaneously in a market characterized by volatile demand. Let $q_i$ be the quantity supplied by firm *i*, $i = 1, 2$. In high demand conditions the inverse demand curve is: $P(Q) = 24 – Q$, for $0 \leq Q \leq 24$, 0 otherwise, where *P* is market price and *Q* is total quantity supplied ($Q = q_1 + q_2$). In low demand conditions the inverse demand curve is: $P(Q) = 12 – Q$, for $0 \leq Q \leq 12$, 0 otherwise. Assume that each demand condition is equally likely. For simplicity, assume there are no costs of production.

   a) Firm 1 hires Greenspan Associates, a consulting firm able to perfectly predict market demand conditions, prior to the two firms selecting quantities. Solve for Bayesian Nash equilibrium in the case where it is common knowledge that firm 1 knows demand conditions but firm 2 does not.
   b) Suppose that firm 1 can commit to release information about demand conditions to firm 2 *prior* to knowing the results of the Greenspan Associates Report. In a perfect Bayesian equilibrium, would firm 1 release information or keep information private? Solve for perfect Bayesian equilibrium in this game where at stage 1, firm 1 makes a choice about information release (before knowing demand conditions), and in stage 2 firms compete by simultaneously setting quantities.
   c) Now suppose that firm 1 could choose to release information *after* it learns the results of the Greenspan Associates Report. In a perfect Bayesian equilibrium, would firm 1 choose to release information after learning either that demand is high or that demand is low? Solve for perfect Bayesian equilibrium in this game where at stage 1, firm 1 makes a choice about information release (after knowing demand conditions), and in stage 2 firms compete by simultaneously setting quantities.
   d) Finally, suppose that firms compete in two periods and that demand conditions are perfectly correlated over the two periods. Firm 1 knows demand conditions and firm 2 does not. In period 1, firm 1 reveals information by playing a separating strategy. In period 1, firm 1 does not reveal information by playing a pooling strategy. Assume that the pooling strategy for firm 1 in period 1 is to play the expected quantity, i.e., the amount produced by firm 2 in part (a). In this case, would firm 1 play the pooling or the separating strategy in period 1?

**Answer:**

a. For firm 1, the objective function is: $Max\ (a^w - q_1^w - q_2)q_1^w$, for $w = H, L$, where $a^H = 24, a^L = 12$. For firm 2 the objective function is: $Max\ 0.5(24 - q_1^H - q_2)q_2 + 0.5(12 - q_1^L - q_2)q_2$. Best response functions are: $24 - 2q_1^H - q_2 = 0,\ 12 - 2q_1^L - q_2 = 0,\ 18 - E(q_1) - 2q_2 = 0.$ Solving for Bayesian Nash equilibrium yields: $q_1^H = 9, q_1^L = 3, q_2 = 6.$

b. If firm 1 releases information, then both firms know what demand is prior to choosing quantity. In this case, quantity choices are given by: $Max\ (a^w - q_i^w - q_j^w)q_i^w$, for $w = H, L.$ Solving for

Nash equilibrium quantities: we find that: $q_1^H = q_2^H = 8,\ q_1^L = q_2^L = 4$. By releasing information, firm 1 would earn expected profits of: 0.5(24-8-8)8+0.5(12-4-4)4=0.5(64+16) = 40. By not releasing information, firm 1 would expect to earn: 0.5(24-9-6)9+0.5(12-3-4)3=0.5(81+9) = 45. In a perfect Bayesian equilibrium, firm 1 would choose not to reveal information, and quantity choices in stage 2 would be as shown in part (a).

c. Consider first what happens when firm 1 learns that demand conditions are low. If firm 1 reveals information about low demand conditions, then firm 2 would produce 4 rather than 6, and firm 1 would be able to earn profits of 16 rather than 9. Firm 1 will choose to reveal information about low demand conditions. Now consider what happens when firm 1 learns that demand conditions are high. Firm 1 would like to not reveal information about high demand conditions, because then firm 2 would produce 8 rather than 6, and firm 1 would earn profit of 64 rather than 81. However, by telling firm 2 that demand conditions are low when they are low, and not revealing information about demand when conditions are high, firm 2 will know all about demand conditions. The perfect Bayesian equilibrium here is that demand information is revealed in stage 1 and quantity choices are $q_1^H = q_2^H = 8,\ q_1^L = q_2^L = 4$.

d. For simplicity, assume there is no discounting. By pooling, firm 1 must produce 6 units in period 1 no matter what the demand conditions are. By doing so, firm 2 will not learn demand conditions and the outcome given in part (a) will occur in period 2. By separating, firm 2 learns demand conditions. Here the outcome is as in part (a) in period 1 and the outcome is as in part (b) in period 2.

High demand conditions

- Pooling: Firm 1 expects to earn (24-6-6)6 + (24-9-6)9 = 72 + 81 = 153
- Separating: Firm 1 expects to earn (24-9-6)9 + (24-8-8)8 = 81 + 64 = 145

Low demand conditions

- Pooling: Firm 1 expects to earn (12-6-6)6 + (12-3-6)3 = 0 + 9 = 9
- Separating: Firm 1 expects to earn (12-3-6)3 + (12-4-4)4 = 9 + 16 = 25

Only if both types would prefer to pool will pooling occur. Because firm 1 would rather separate when demand conditions are low, then firm 1 will play the separating strategy. But if firm 1 of type *L* plays the separating strategy $(q_1^L = 3)$, the firm 1 of type H is revealed to be type H if it plays $q_1^H = 6.$ It would be better in this case to fully separate. Therefore, the perfect Bayesian equilibrium in this case is for the separating equilibrium strategy to be played in the first period: $q_1^H = 9, q_1^L = 3, q_2 = 6$, and the full information strategies to be played in the second period: $q_1^H = q_2^H = 8,\ q_1^L = q_2^L = 4$.

In general, one can show there is a unique separating perfect Bayesian equilibrium to this game. The low type prefers to separate. By playing the optimal separating strategy in period 1 ($q_1^L = 3$), the high type would find it more profitable to play its optimal separating strategy ($q_1^H = 9$) rather than mimic the low type by playing $q_1^H = 3$.