Game Theory
APEC 8205

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Final Exam

Please answer all of the following questions. Please be clear and concise in your answers (define terms and label figures). This exam is open mind but closed book. Time limit: 3 hours. Think before writing. Relax. Do well.

1. Suppose there are two firms that produce different, but related, products. Let x be the output of firm 1 and y be the output of firm 2. Marginal costs are assumed to be zero. Demand for firm 1's product is $p_1 = a_1 - by - x$, while demand for firm 2's product is $p_2 = a_2 - bx - y$ where b captures the degree of substitutability between the two products. For example, if $b > 0$, the two products are substitutes, while for $b < 0$, the two products are complements. Suppose firm 2 has incomplete information regarding the level of firm 1's demand (i.e. on a_1). Everything else about the demand conditions is known to both players. Let m and $s^2 = E(a_1^2) - m^2$ characterize firm 2's beliefs regarding the mean and variance of a_1. Firm 1 knows these beliefs.

   a) Find the Bayesian Nash equilibrium output for both firms assuming quantities are chosen simultaneously and parameter values produce an interior solution.
   b) Holding the variance constant, how does an increase in firm 2's mean belief affect the equilibrium quantities? Discuss the intuitive appeal of your result.

ANSWER

a) Firm 1's payoff is $p_1 = (a_1 - by - x)x$. The first order condition is: $a_1 - by - 2x = 0$ and best response function $x(a_1) = (a_1 - by)/2$.

Firm 2's payoff is $p_2 = E[(a_2 - bx - y)y]$. The first order condition is: $a_2 - E(bx(a_1)) - 2y = 0$ and best response function $y = (a_2 - bE(x(a_1))/2$.

Solve for Bayesian Nash equilibrium:

$$y^* = \frac{a_2 - bE\left(\frac{a_1 - by^*}{2}\right)}{2} = \frac{2a_2 - bm - by^*}{4}$$

$$y^* = \frac{2a_2 - bm}{4 - b}$$

$$x^* = \frac{a_1(4-b) - 2a_2 b + b^2 m}{8 - 2b}$$

b) $\frac{\partial y^*}{\partial m} = \frac{-b}{4-b} < 0$ and $\frac{\partial x^*}{\partial m} = \frac{b^2}{2(4-b)} > 0$

Therefore, firm 2 decreases its effort and firm 1 increases its effort if the two goods are substitutes. If they are complements, both firm will increase effort. Does this make sense? Yes. If the two goods are substitutes, when firm 1 has a higher level of demand it is optimal for it to produce more, which reduces the demand for firm 2. Firm 2's best response is to reduce its supply in light of the reduced demand. Alternatively, if the goods are complements, an increase in firm 1's demand will increase firm 2's demand. It is then optimal for Firm 2 to increase its output in response to its increased demand.

2. Consider the following simple game in which all players $i = 1, 2, \ldots, N$, simultaneously choose between strategy *A* and strategy *B*. The game is symmetric so that only the payoffs to a single player need to be represented. The payoffs given in the table below are for the row player. Let *p* be the proportion of players that play strategy *A* and (1 - *p*) be the proportion of players that play strategy *B*. Suppose that players do not necessarily start out at equilibrium but that their strategy evolves over time based on the relative expected payoffs to playing different strategies. For this problem, assume that the evolution of strategies is given by continuous-time replicator dynamics:

$$dp/dt = p(\Pi(p) - \overline{\Pi}(p)),$$

where $\Pi(p)$ is the expected payoff from playing strategy *A* given *p*, and $\overline{\Pi}(p)$ is the expected payoff for all players given *p*.

	A	B
A	3	4
B	0	9

a) Solve the equilibrium dynamics for this game, i.e., solve for the equation that determines *dp*/*dt*.
b) What are the stationary points of *dp*/*dt* (i.e., equilibrium values of *p*)? Which of these stationary points are stable? Explain.
c) Suppose the initial value of *p* was 0.4. What is the value of *p* to which the system would evolve?
d) Replicator dynamics have been criticized as leaving out important elements of learning and behavior by strategic players in games. Briefly discuss the major limitations of replicator dynamics and explain an alternative to approach that addresses at least one major limitation.

ANSWER

a)

$$\begin{aligned} dp/dt &= p(\Pi(p) - \overline{\Pi}(p)) \\ &= p([3p + 4(1-p)] - \{p[3p + 4(1-p)] + (1-p)[9(1-p)]\}) \\ &= p(1-p)(3p + 4(1-p) - 9(1-p)) \\ &= p(1-p)(8p - 5). \end{aligned}$$

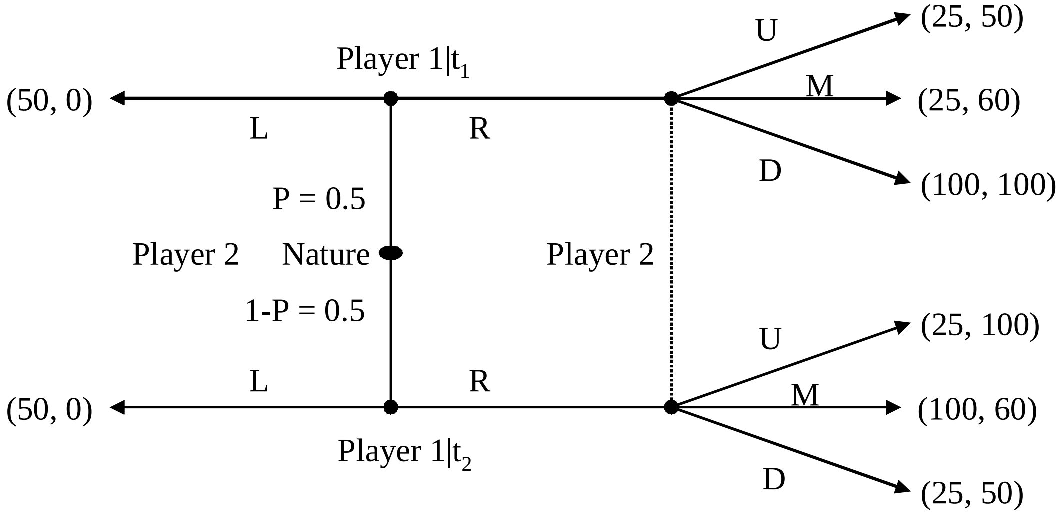
b. *dp*/*dt* = 0, for *p* = 0, *p* = 1, *p* = 5/8. For 0 < *p* < 5/8, *dp*/*dt* < 0. For 5/8 < *p* < 1, *dp*/*dt* > 0. So for *p* < 5/8, *p* will approach 0. For *p* > 5/8, *p* will approach 1. Both *p* = 0 and *p* = 1 are stable; *p* = 5/8 is unstable.

c. For an initial *p* = 0.4, *dp*/*dt* < 0, and *p* will evolve toward 0. The equilibrium prediction for this game is *p* = 0, i.e., players will play strategy *B*.

d. Replicator dynamics is backward looking. It does not anticipate strategy choices and take actions accordingly (lack of sophistication). Replicator dynamics can get locked into local dynamics and make predictions that most would think are unlikely. Convergence issues….

3. Answer the following questions for the signaling game shown in the figure below. Player 1 moves first and can choose L to end the game or R to let player 2 move. If player 1 chooses R, player 2 can choose U, M, or D. Player 1 can be one of two types, t_1 with probability P = 0.3 or t_2 with probability 1 – P = 0.7. Player 2 does not know player 1's type, but does observe whether player 1 chooses L or R.

   a) Find all pure strategy perfect Bayesian equilibria for this signaling game.
   b) Which of the perfect Bayesian equilibria from part (a) satisfy the Intuitive Criterion?



ANSWER

a) Lets begin with finding 2's best response given the probability of t_1 when R is chosen, m. U is always better than M because $50m + 100(1-m) >(<) 50m + 70(1-m)$. M is greater than D when $50m + 70(1-m) >(<) 100m + 50(1-m)$ or when $2/7 >(<) m$. U is greater than D when $50m + 100(1-m) >(<) 100m + 50(1-m)$ or when $1/2 >(<) m$. Therefore, player 2's best response is D for $m \geq ½$ and U for $m \leq ½$. Note that M is never a best response for player 2.

Now that we know player 2's best response, we can consider the four possible types of pure strategy equilibria.

First, suppose both types of 1 pool on L. L is a best response for both only if 2 plays U or if $m \leq ½$. Bayes rule does not apply in this case so we can not rule out $m \leq ½$. {(L,L), U|R, $m \leq ½$} is a perfect Bayesian equilibrium.

Second, suppose both types of 1 pool on R. But R is a best response for a t_1 1 only if 2 plays D, and R is a best response for a t_2 1 only if 2 plays M. Therefore, there is no strategy for 2 that makes R a mutual best response for both types of player 1.

Third, suppose a t_1 chooses R and a t_2 chooses L. Bayes rule implies $m = 1$. If $m = 1$, 2's best response is D, which R is a best response to for t_1 and L is a best response for for t_2. Therefore, {(R,L), D|R, $m = 1$} is a perfect Bayesian equilibrium.

Fourth, suppose a t_1 chooses L and a t_2 chooses R. Bayes rule implies $m = 0$. If $m = 0$, 2's best response is U, which makes L instead of R a best response for t_2. Therefore, this cannot be perfect Bayesian equilibrium.

b) Since the Intuitive Criterion disciplines beliefs off the equilibrium path, we know it can only work to get rid of {(L,L), U|R, $m \leq ½$}. But how does it work. A t_1's best response is R if 2 chooses D and D is a best response for 2 when $m \geq ½$. A t_2's best response is R if 2 chooses M, but M is never a best response for 2. Therefore, if 2 sees R, the Intuitive Criterion dictates 2 beliefs $m = 1$ or that player 1 is of type t_1.

4. In each round of the "investment game," Jane Veryritch decides whether or not to invest with the H&P Stock Market Investment Company. H&P can be one of two types: i) a fraudulent company, or ii) an honest company. H&P knows its type. Jane has an initial probability of p that the company is honest. A fraudulent type cannot invest in stocks but only in a low-yielding non-risky asset. An honest company will invest Jane's money in the stock market. In the stock market it can either be a good year or a bad year. Suppose there is an 80% probability that a round will be a good year and a 20% probability that a round will be a bad year.

   To keep matters (relatively) simple suppose that the actions of the two players in each stage are as follows:

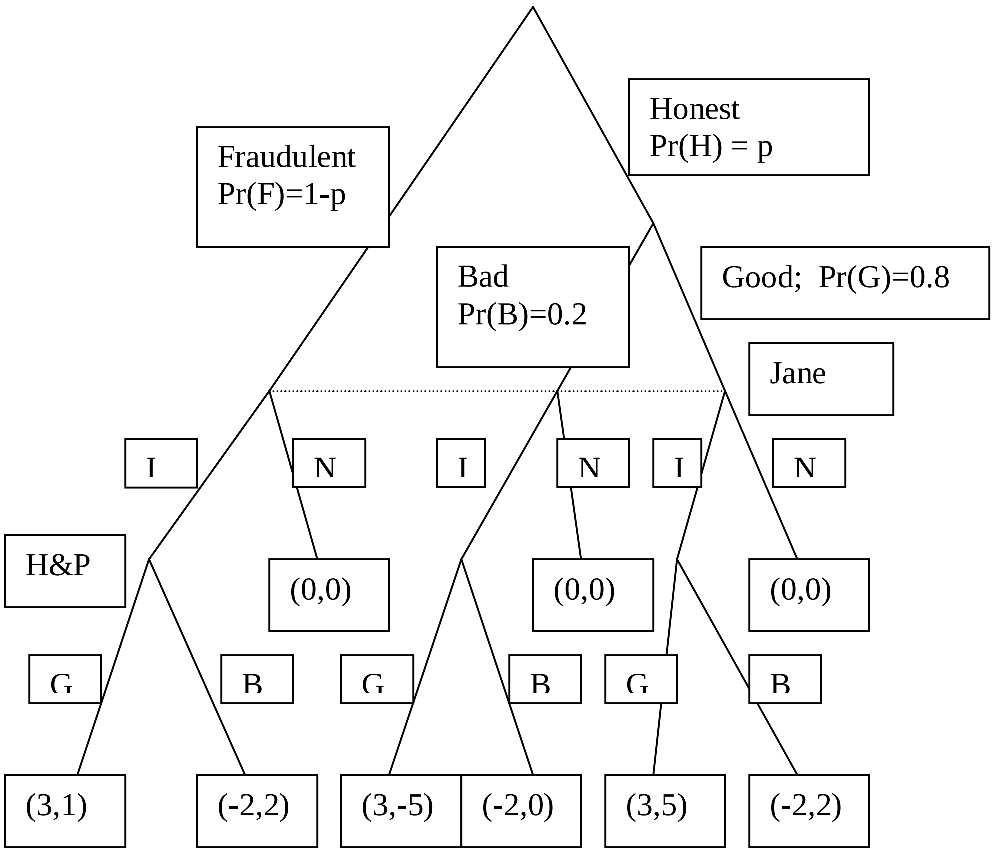
   - Jane: invest (I) or not invest (N)
   - H&P: announce good year (G) or bad year (B).

   If Jane does not invest, both Jane and H&P get payoffs of \$0. If Jane invests and H&P announces a good year, then Jane gets a payoff of \$3. If Jane invests and H&P announces a bad year, Jane loses \$2. With investment and announcement of a good year when it is a good year, an honest H&P gets a payoff of \$5. With investment and announcement of a good year when it really is a bad year, an honest H&P loses \$5. With investment and announcement of a good year, a fraudulent H&P gains \$1. With investment and announcement of a bad year when it is really a good year, an honest H&P gets a payoff of \$2 (even though they are rolling in money they feel somewhat bad about cheating Jane). With investment and announcement of a bad year when it is a bad year, an honest H&P gets a payoff of \$0. With investment and announcement of a bad year, a fraudulent H&P runs off with the money and gets a payoff of \$2. (Note: because Jane is really very rich, all figures are in millions of dollars.)

   All of the information given above is summarized in the game tree for the investment game if played for only a single round.

   Assume that both Jane and H&P are risk neutral and that they do not discount future payoffs.

   a. Suppose this game is played for a single round. Solve for perfect Bayesian equilibrium for various values of p. What is the critical value of p that would make Jane indifferent between investing and not investing?
   b. Now suppose that the investment game will be played for two rounds and let $p = 0.25$. Is there a perfect Bayesian equilibrium where both the honest and fraudulent types announce good in the first round regardless of whether it is a good year or not? If so, define the complete set of equilibrium strategies. If not, explain why.
   c. As in part (b), suppose that the investment game will be played for two rounds and let $p = 0.25$. Now assume that the honest type always announces honestly (announces good for a good year and bad for a bad year) and that the fraudulent type announces good with probability q (regardless of type of year). What value of q in the first round would make Jane indifferent between investing in the second round upon observing an announcement of good?
   d. Is there a perfect Bayesian equilibrium where H&P plays the strategy outlined in part (c) in the first round? If so, define the complete set of equilibrium strategies. If not, explain why.


Honest
Pr(H) = p
Fraudulent
Pr(F)=1-p
Bad
Pr(B)=0.2
Good; Pr(G)=0.8
Jane
I
N
I
N
I
N
H&P
(0,0)
(0,0)
(0,0)
G
B
G
B
G
B
(3,1)
(-2,2)
(3,-5)
(-2,0)
(3,5)
(-2,2)

ANSWERS

a. H&P will announce good year when it is honest and it is a good year, otherwise it will announce bad year. For Jane, the expected payoff to investing is:

$$3(.8)p - 2(.2)p - 2(1\text{-}p) = 4p - 2.$$

For $p < 0.5$, Jane will not invest. For $p > 0.5$, Jane will invest. The critical value of p that makes Jane indifferent between investing and not investing is 0.5.

b. No. If all types play G then the Pr(H|G) = p = 0.25. But then Jane will not invest in the second period. If Jane will not invest upon seeing G. But then if there is a bad year or H&G is the fraudulent type, they should announce B rather than G.

c. For Jane to be indifferent in the second round, it must be that Pr(H|G)=0.5. Using Bayes' Rule we have:

$$\Pr(H \mid G) = \frac{\Pr(G \mid H)\Pr(H)}{\Pr(G \mid H)\Pr(H) + \Pr(G \mid F)\Pr(F)}$$
$$= \frac{0.8 * 0.25}{0.8 * 0.25 + q * 0.75}$$

Set Pr(H|G)=0.5 and solve for q:

$$\frac{0.8 * 0.25}{0.8 * 0.25 + q * 0.75} = 0.5$$
$$q = 4/15$$

d. Yes. Suppose that H&P plays as described in part (c) in the first round and plays G if honest and it is a good year in round 2, but otherwise plays B in round 2. In round 1, Jane will be indifferent between investing and not investing because her expected payoff from investment in the first round is:

$$3(.8)*0.25 - 2(.2)*0.25 + 3*0.75*(4/15) - 2*0.75*(11/15)$$
$$= 0.6-0.1+0.6-1.1 = 0$$

If Jane does not invest in the first round, then the Pr(H) in the second round is equal to p = 0.25. Jane will not enter in the second round so that she earns $0 overall. If Jane does invest in round 1, then she will update according to the answer in part (c) if H&P announces G. In the second period she expects to earn $0 from investing and $0 from not investing. If H&P announces B, the resulting updated probability is:

$$\Pr(H \mid B) = \frac{\Pr(B \mid H)\Pr(H)}{\Pr(B \mid H)\Pr(H) + \Pr(B \mid F)\Pr(F)}$$
$$= \frac{0.2 * 0.25}{0.2 * 0.25 + (11/15) * 0.75}$$
$$= 1/12$$

Since this is less than 0.5, Jane will not invest after an announcement of B in round 1 and earn a payoff of $0. Whether or not Jane invests in the first round, or invests in the second round after seeing G, she expects to get a payoff of $0.

At this point, we have checked that Jane updates according to Bayes' Rule and that she plays a best response given her beliefs and the strategy of H&P. We next need to check that H&P is choosing a best response in all cases given Jane's strategy. Note: we haven't yet specified Jane's probability of investing in the first round or in the second round after seeing G in the first round. Suppose Jane mixes with some probability in the first round and sets the probability of investing after seeing G to be equal to *x*.

i) Honest type and good year in round 1: by announcing G, H&P gets a payoff of $5 and maintains the possibility of investment in the second round, which generates positive expected returns. If H&P were to announce B instead, they would get $2 in round 1 and $0 in round 2. Announcing G is a best response in the first round.

ii) Honest type and bad year in round 1: by announcing B, H&P gets a payoff of $0 and gets $0 in the second round (no investment). If H&P were to announce G instead, they would get -$5 in round 1 and an expected value of $4*$x$ in round 2. This will necessarily be less than $5 so that announcing B is a best response in the first round.

iii) Fraudulent type: the fraudulent type must be indifferent between announcing G and announcing B. The expected payoff from announcing B is $2 in round 1 and $0 in round 2. The expected payoff from announcing G is $1 in round 1 and $2*$x$ in round 2. If $x = 0.5$, then the fraudulent type will be indifferent be announcing G and B in round 1.

Since both players are playing best responses in all circumstances and updating satisfies Bayes' Rule, this set of strategies constitutes a perfect Bayesian equilibrium.