# Game Theory
# APEC 8205

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## Final Exam

Please answer all of the following questions. Please be clear and concise in your answers (define terms and label figures). This exam is open mind but closed book. Time limit: 3 hours. Think before writing. Relax. Do well.

1. Suppose there are two firms that produce similar, but not identical products. Let $x$ be the output of firm 1 and $y$ be the output of firm 2. Marginal costs are assumed to be zero. Demand for firm 1's product is $p_1 = a_1 - by - x$, while demand for firm 2's product is $p_2 = a_2 - bx - y$ where $b$ captures the degree of substitutability between the two products. For example, if $b > 0$, the two products are substitutes, while for $b < 0$, the two products are complements. Suppose firm 1 knows $b$, but not firm 2. Let $m$ and $s^2 = E(b^2) - m^2$ represent firm 2's beliefs regarding the mean and variance of $b$. Firm 1 knows these beliefs.

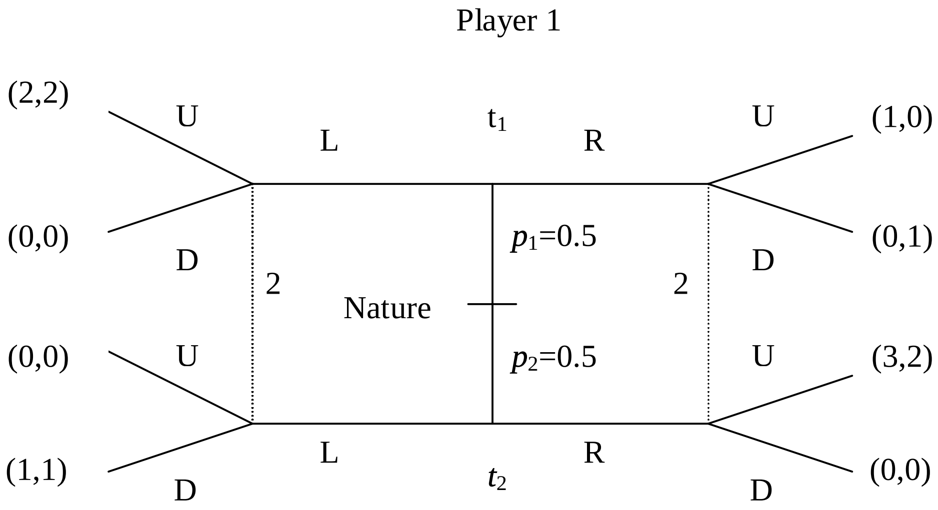
a) Find the Bayesian Nash equilibrium output for both firms assuming quantities are chosen simultaneously and parameter values produce an interior solution.
b) Holding the variance constant, how does an increase in firm 2's mean belief affect the equilibrium quantities? Discuss the intuitive appeal of your result.
c) Holding the mean constant, how does an increase in firm 2's uncertainty (e.g. increase in the variance of firm 2's belief) affect the equilibrium quantities? Discuss the intuitive appeal of your result.

2. Show what the dynamics of the game will be assuming replicator dynamics for the 2x2 symmetric game shown below. Is there a unique equilibrium prediction of where the game will evolve or does it depend upon initial conditions?

|   | A | B |
|---|---|---|
| A | 3 | 0 |
| B | 1 | 2 |

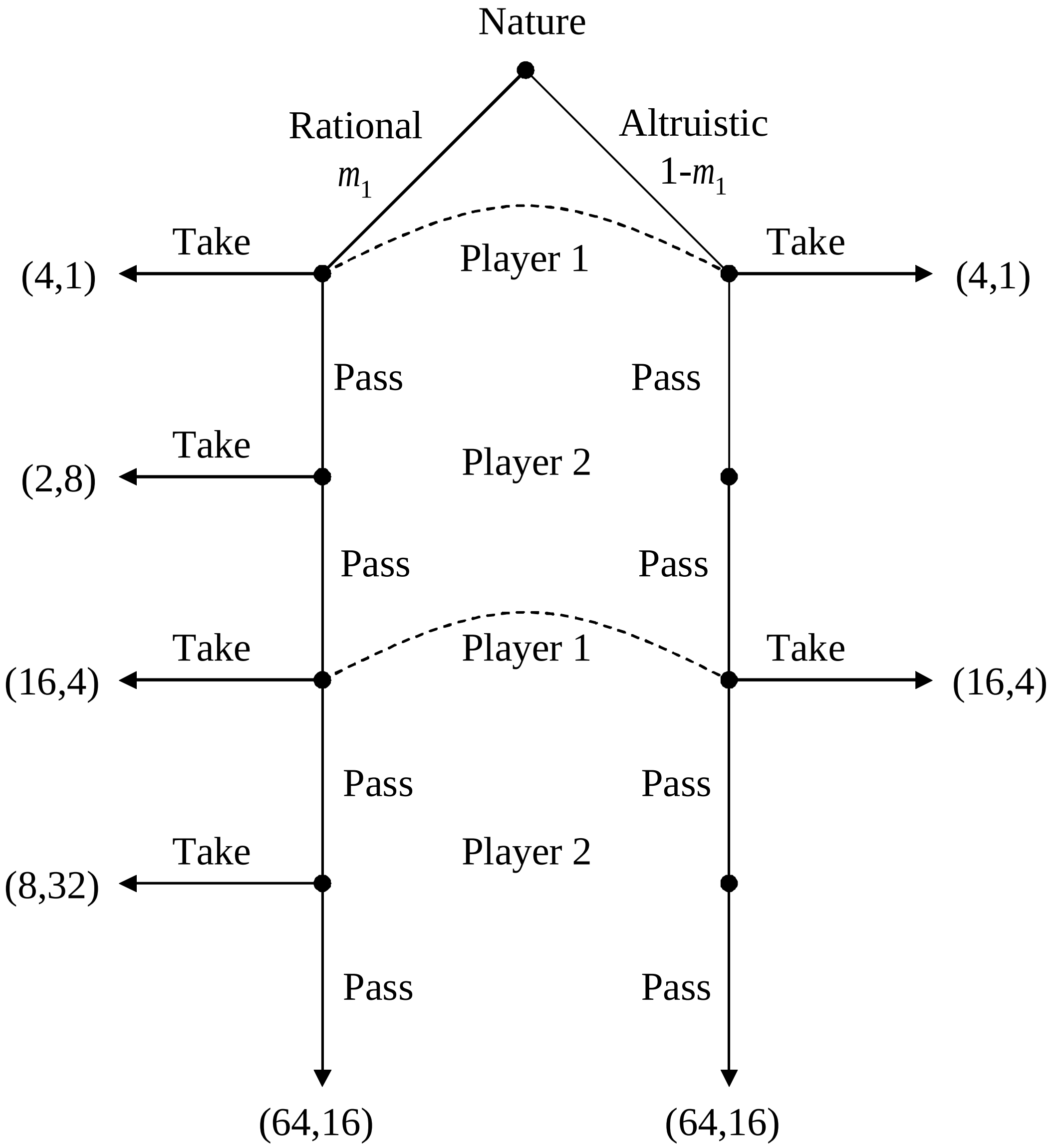
3. Answer the following questions for the signaling game shown in the figure below. Player 1 moves first and can go L or R. Player 2 then moves and can go U or D. Player 1 can be of two types, $t_1$ and $t_2$. Player 2 does not know player 1's type but does observe whether player 1 chooses L or R.

a) Find all pure strategy perfect Bayesian equilibria for the signaling game.
b) Which of the perfect Bayesian equilibria from part (a) satisfy the Intuitive Criterion?
c) Now suppose the game is changed so that player 2 always knows the type for player 1 when player 1 chooses Left. Find all pure strategy perfect Bayesian equilibria for this game.



4. Consider the incomplete information variation of the two player, four move centipede game in the figure below. The first and second players' payoffs are the first and second numbers in parentheses. Nature starts the game by choosing whether or not player 2 is Rational or Altruistic with probability $m_1$ and $1 - m_1$. Then player 2's type is revealed to player 2, but not player 1. The players then play the four move centipede game with player 1 moving first. Note an Altruistic player 2 will always pass. Alternatively, Player 1 and the Rational player 2 try to maximize their expected payoffs.

a) Find the subgame perfect equilibrium for this game when $m_1 = 1$ (i.e. there are no Altruistic types).
b) Find the perfect Bayesian equilibrium for this game when $m_1 = ½$.
c) How well do the predictions of each of these games explain experimental results using centipede games?



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## Answers to the Final Exam

1. (16 points: part (a) – 8 points, part (b) – 4 points, part (c) = 4 points)

   a. Firm 1's payoff is $p_1 = (a_1 - by - x)x$. The first order condition is: $a_1 - by - 2x = 0$ and best response function $x = (a_1 - by)/2$.

   Firm 2's payoff is $p_2 = E[(a_2 - bx - y)y]$. The first order condition is: $a_2 - E(bx(b)) - 2y = 0$ and best response function $y = (a_2 - E(bx(b))/2$.

   Solve for perfect Bayesian equilibrium:

$$y^* = \frac{a_2 - E\left(b\frac{a_1 - by^*}{2}\right)}{2} = \frac{2a_2 - a_1 E(b) + y^* E(b^2)}{4}$$

$$y^* = \frac{2a_2 - a_1 E(b)}{4 - E(b^2)} = \frac{2a_2 - a_1 m}{4 - s^2 - m^2}$$

$$x^* = \frac{a_1 - b\frac{2a_2 - a_1 m}{4 - s^2 - m^2}}{2} = \frac{a_1}{2} - \frac{b(2a_2 - a_1 m)}{2(4 - s^2 - m^2)}$$

   b. $$\frac{\partial y^*}{\partial m} = \frac{-a_1(4 - s^2 + m^2) + 2m(2a_2 - a_1 m)}{\left(4 - s^2 - m^2\right)^2}$$

   The sign of this expression is ambiguous as the first term in the numerator is negative and the second term is positive (for an interior solution). There are a number of effects, which do not all move in the same direction. Changing expectations about *b* (but not changing *b* itself) changes the expectation by firm 2 about the nature of the demand curve and competition with firm 1. (Confession: when we originally did this problem, we made a sign error and thought we could sign this expression.)

   For firm 1: $\frac{\partial x^*}{\partial m} = \frac{-b}{2}\frac{\partial y^*}{\partial m}$ . Note that $sign(\frac{\partial x^*}{\partial m}) = sign(\frac{\partial y^*}{\partial m})$ for $b < 0$, but they move in the opposite direction for $b > 0$.

c. $\frac{\partial y^*}{\partial s^2} = \frac{y^*}{4 - s^2 + m^2} > 0$ and $\frac{\partial x^*}{\partial s^2} = \frac{-b}{2}\frac{\partial y^*}{\partial s^2}$ <(>) 0 if $b$ >(<) 0.

Firm 2 tends to expand output more when it thinks conditions are favorable and than it decreases output when it thinks conditions are unfavorable so that output expands with increased variance. If firm 1's output is a substitute for firm 2 ($b > 0$), it will move in the opposite direction of firm 2's output. However, if the products are complements ($b < 0$), firm 1's output will move in the same direction as firm 2's output.

2. (10 points)

Let $x$ be the probability of playing strategy *A* and (1-$x$) be the population probability of playing strategy *B*. The payoff to playing strategy *A* is: $3x$. The payoff to playing strategy *B* is: $1*x + 2(1-x) = 2 - x$.

In replicator dynamics, a probability of playing a strategy will increase when the expected payoff to playing the strategy is higher than the average expected payoff over all strategies:

$$\begin{aligned}
\frac{dx}{dt} &= x(p_x - \bar{p}) \\
&= x(3x - (x3x + (1-x)(2-x)) \\
&= x(3x(1-x) - (1-x)(2-x)) \\
&= x(1-x)(4x-2) \\
&= 4x(1-x)(x-1/2)
\end{aligned}$$

For 0 < x < ½, $dx/dt < 0$. For ½ < x < 1, $dx/dt > 0$. Therefore, if $x$ starts above ½ the dynamics will take it to $x = 1$ (which is a pure strategy Nash equilibrium). If $x$ starts below ½ the dynamics will take it to $x = 0$ (which is also a pure strategy Nash equilibrium). If $x$ = ½ (a mixed strategy Nash equilibrium), it will stay there. But any slight perturbation will inevitably take $x$ to 0 or 1.

3. (20 points: part (a) – 8 points, part (b) – 6 points, part (c) – 6 points)

a. There are four possible pure strategy profiles to check: pooling on *L*, pooling on *R*, separating where $t_1$ plays *L* and $t_2$ plays *R*, and separating where $t_1$ plays *L* and $t_2$ plays *R*. Let $m_L = \Pr(t_1 \mid L)$ and $m_R = \Pr(t_1 \mid R)$.

*Pooling on L*: In this case, Bayes' Rule implies that $m_L = 0.5$ and $m_R$ can be anything. Given $m_L = 0.5$, the best response of the receiver given *L* is *U*. This gives a payoff of 2 for $t_1$ and 0 for $t_2$. No matter how the receiver responds upon observing *R*, $t_1$ will always do better by playing *L* than *R*.

For $t_2$, playing *L* is a best response only if the receiver will play *D* upon seeing *R*. The receiver will play *D* if $m_R \geq 2/3$. Therefore, pooling on *L* is a perfect Bayesian equilibrium. We can summarize the perfect Bayesian equilibrium as follows: {$t_1$ strategy, $t_2$ strategy; receiver strategy if *L*, receiver strategy if *R*; $m_L$, $m_R$}
{*L*, *L*; *U*, *D*; $m_L = 0.5$, $m_R \geq 2/3$}.

*Pooling on R*: In this case, Bayes' Rule implies that $m_R = 0.5$ and $m_L$ can be anything. Given $m_R = 0.5$, the best response of the receiver given *R* is *U*. This gives a payoff of 1 for $t_1$ and 3 for $t_2$. No matter how the receiver responds upon observing *L*, $t_2$ will always do better by playing *R* than *L*. For $t_2$, playing *R* is a best response only if the receiver will play *D* upon seeing *L*. The receiver will play *D* if $m_L \leq 1/3$. Therefore, pooling on *R* is a perfect Bayesian equilibrium. We can summarize the perfect Bayesian equilibrium as follows: {*R*, *R*; *D*, *U*; $m_L \leq 1/3$, $m_R = 0.5$}.

*Separating where $t_1$ plays L and $t_2$ plays R*: In this case, Bayes' Rule implies that $m_L = 1$ and $m_R = 0$. Given $m_L = 1$, the best response of the receiver given *L* is *U*. Given $m_R = 0$, the best response of the receiver given *R* is *U*. This gives a payoff of 2 for $t_1$ when playing *L*. If instead $t_1$ plays *R*, $t_1$ will receive a payoff of 1. Playing *R* gives $t_2$ a payoff of 3, which is better than getting 0 by playing *L*. Therefore, separating where $t_1$ plays *L* and $t_2$ plays *R* is a perfect Bayesian equilibrium. We can summarize the perfect Bayesian equilibrium as follows:
{*L*, *R*; *U*, *D*; $m_L = 1$, $m_R = 0$}.

*Separating where $t_1$ plays R and $t_2$ plays L*: In this case, Bayes' Rule implies that $m_L = 0$ and $m_R = 1$. Given $m_L = 0$, the best response of the receiver given *L* is *D*. Given $m_R = 0$, the best response of the receiver given *R* is *D*. This gives a payoff of 0 for $t_1$ when playing *R*. If instead $t_1$ plays *L*, $t_1$ will receive a payoff of 0. Playing *L* gives $t_2$ a payoff of 1, which is better than getting 0 by playing *R*. Therefore, separating where $t_1$ plays *L* and $t_2$ plays *R* is a perfect Bayesian equilibrium. We can summarize the perfect Bayesian equilibrium as follows:
{*R*, *L*; *D*, *D*; $m_L = 0$, $m_R = 1$}.

b. The Intuitive Criterion places restrictions on off-equilibrium path beliefs. For separating equilibria in this game, all strategies are played by some type so there are no off-equilibrium beliefs. Therefore, both perfect Bayesian separating equilibria will satisfy the Intuitive Criterion.

In pooling equilibria, there are off-equilibrium path beliefs that may be restricted by the Intuitive Criterion. The Intuitive Criterion states that $p(t|a) = 0$ if playing action *a* is equilibrium dominated for type *t* but is not equilibrium dominated for some other type. An action is equilibrium dominated if the equilibrium payoff strictly exceeds the payoffs for an action under all possible choices by rival players.

*Pooling on L*: In this perfect Bayesian equilibrium, type $t_1$ receives a payoff of 2, type $t_2$ receives a payoff of 0. Playing $R$ is equilibrium dominated for $t_1$ but not for $t_2$. Therefore, the receiver should set $m_R = 0$. The best response given these beliefs is $U$. But then $t_2$ should play $R$ instead of $L$. Pooling on $L$ does not satisfy the Intuitive Criterion.

*Pooling on R*: In this perfect Bayesian equilibrium, type $t_1$ receives a payoff of 1, type $t_2$ receives a payoff of 3. Playing $L$ is equilibrium dominated for $t_2$ but not for $t_1$. Therefore, the receiver should set $m_L = 1$. The best response given these beliefs is $U$. But then $t_1$ should play $L$ instead of $R$. Pooling on $R$ does not satisfy the Intuitive Criterion.

The only equilibria that satisfy the Intuitive Criteria are the two separating equilibria.

c. With knowledge of type when $L$ is played, the best response of the receiver is to play $U$ if they see that $t_1$ played $L$, and to play $D$ if they see that $t_2$ played $L$. For $t_1$, they are always better off playing $L$. So, the only possible pure strategy perfect Bayesian equilibria are where both types play $L$ and separating where $t_1$ plays $L$ and $t_2$ plays $R$. Using similar logic to what was done in part (a), it is straightforward to show that both of the following are perfect Bayesian equilibria:

{$L$, $L$; $U$ if $t_1$ plays $L$, $D$ if $t_2$ plays $L$, $D$ if $R$, $m_R \geq 2/3$}
{L, $R$; $U$ if $t_1$ plays $L$, $D$ if $t_2$ plays $L$, $U$ if $R$, $m_R = 0$}

4. (18 points: part (a) – 4 points, part (b) – 8 points, part (c) – 6 points)

a. In move 4, Player 2 should choose Take because 32 > 16. Given player 2 chooses take in move 4, player 1 should choose Take in move 3 because 16 > 8. Given player 1 chooses Take in move 3, player 2 should choose Take in move 2 because 8 > 4. Finally, given player 2 chooses Take in move 2, player 1 end the game immediately by choosing Take in move 1 because 4 > 2. The subgame perfect equilibrium strategies for both players is to choose Take each time it is their move.

b. **Definitions**
$s^2_i|R$ : probability a rational player 2 chooses Pass at information set $i$
$s^2_i|A$=1: probability an altruistic player 2 chooses Pass at information set $i$.
$s^1_i$ : probability player 1 chooses Pass at information set $i$.
$m_i$: player 1's beliefs at information set $i$ about the probability of player 2 being a rational type.

**Move 4**

Player 2’s sequentially rational choices are $s^2_2|A = 1$ because an altruistic type always chooses Pass and $s^2_2|R = 0$ because 32 > 16.

**Move 3**

Player 1’s expected payoff given player 2 sequentially rational response move 4 is

$s^1_2(8m_2 + 64(1 - m_2)) + 16(1 - s^1_2) = s^1_2(48 - 56m_2) + 16.$

Therefore,

$s^1_2 = 1$ if $m_2 < 48/56 = 6/7$
$s^1_2 = [0,1]$ if $m_2 = 6/7$
$s^1_2 = 0$ if $m_2 > 6/7$

Bayes’ rule implies

$$m_2 = \frac{(s_1^2 \mid R)m_1}{(s_1^2 \mid R)m_1 + (s_1^2 \mid A)(1 - m_1)} = \frac{(s_1^2 \mid R)m_1}{(s_1^2 \mid R)m_1 + (1 - m_1)}$$

so that
$m_2$ </=/> 6/7 as $s^2_1|R$ </=/> $6(1 - m_1)/m_1$

**Move 2**

An altruistic Player 2’s sequentially rational choice is $s^2_1|A = 1$ because an altruistic type always choose Pass.

A Rational player’s expected payoff is

$32s^2_1|R + 8(1- s^2_1|R) = 24s^2_1|R + 8$ if $s^2_1|R < 6(1 - m_1)/m_1$, so

$s^2_1|R = 1$ with payoff 32, which implies $1 < 6(1 - m_1)/m_1$ or $m_1 < 6/7$.

For $1 > s^2_1|R = 6(1 - m_1)/m_1 > 0$, the rational player must be indifferent between Pass and Take implying

$32s^1_2 + 4(1 - s^1_2) = 8$ or $s^1_2 = 4/28 = 1/7$, for a payoff of 8.

Also note that $1 > 6(1 - m_1)/m_1 > 0$ implies $1 > m_1 > 6/7$.

For $s^2{}_1|R > 6(1 - m_1)/m_1$, the Rational player's payoff is

$4s^2{}_1|R + 8(1 - s^2{}_1|R) = 8 - 4s^2{}_1|R$,

so $s^2{}_1|R = 0$ with payoff 8. Note however that $0 > 6(1 - m_1)/m_1$ implies $m_1 > 1$, which can never be the case. Therefore, if player 1 passes in move 1, it will always be optimal for player 2 to pass with some positive probability in move 2.

**Move 1**

Player 1
We have two scenarios to consider: $1 > m_1 > 6/7$ and $6/7 > m_1 > 0$

First, suppose $6/7 > m_1 > 0$, so $s^2{}_1|R = 1$, which implies $m_2 < 6/7$, so $s^1{}_2 = 1$, $s^2{}_2|R = 0$, and $s^2{}_2|A = 1$. Player 1's expected payoff is then

$s^1{}_1(8m_1 + 64(1 - m_1)) + 4(1 - s^1{}_1) = s^1{}_1(60 - 56m_1) + 4$.

Therefore, $s^1{}_1 = 1$ because $60/56 > m_1$.

Now, suppose $1 > m_1 > 6/7$, so $s^2{}_1|R = 6(1 - m_1)/m_1$, which implies $m_2 = 6/7$, so $s^1{}_2 = 4/28$, $s^2{}_2|R = 0$, and $s^2{}_2|A = 1$. Player 1's expected payoff is then

$s^1{}_1(((8s^1{}_2 + 16(1 - s^1{}_2))s^2{}_1|R + 2(1 - s^2{}_1|R))m_1 + (64s^1{}_2 + 16(1 - s^1{}_2))(1 - m_1))$
$+ 4(1 - s^1{}_1) = s^1{}_1(96 – 98m_1) + 4$

Therefore,

$s^1{}_1 = 1$ if $m_1 < 96/98 = 48/49$
$s^1{}_1 = [0,1]$ if $m_1 = 96/98 = 48/49$
$s^1{}_1 = 0$ if $m_1 > 96/98 = 48/49$

We are now ready to summarize. There are four potential types of equilibria depending on the initial belief $m_1$:

(i) if $1 \geq m_1 \geq 48/49$, $s^1{}_1 = 0$ and the game ends immediately.
(ii) if $m_1 = 48/49$, $s^1{}_1 = (0,1)$, $s^2{}_1|R = 6(1 - m_1)/m_1$, $s^2{}_1|A = 1$, $s^1{}_2 = 4/28$, $s^2{}_2|R = 0$, $s^2{}_2|A = 1$, and $m_2 = 6/7$.
(iii) if $48/49 \geq m_1 \geq 6/7$, $s^1{}_1 = 1$, $s^2{}_1|R = 6(1 - m_1)/m_1$, $s^2{}_1|A = 1$, $s^1{}_2 = 4/28$, $s^2{}_2|R = 0$, $s^2{}_2|A = 1$, and $m_2 = 6/7$.

(iv) if $6/7 \geq m_1 \geq 0$, $s^1{}_1 = 1$, $s^2{}_1|R = 1$, $s^2{}_1|A = 1$, $s^1{}_2 = 1$, $s^2{}_2|R = 0$, $s^2{}_2|A = 1$, and $m_2 < 6/7$.

Therefore, $m_1 = ½$, the perfect Bayesian equilibrium is $s^1{}_1 = 1$, $s^2{}_1|R = 1$, $s^2{}_1|A = 1$, $s^1{}_2 = 1$, $s^2{}_2|R = 0$, $s^2{}_2|A = 1$, and $m_2 < 6/7$.

c. In experimental treatments using the centipede game, researches have found that few subjects start the game by choosing Take (the subgame perfect equilibrium). They also find that the game seldom progresses far enough for the last person to get a move (the perfect Bayesian equilibrium). Therefore, the subgame perfect equilibrium predicts not enough passing, while the perfect Bayesian equilibrium predicts too much passing. However, if we assumed $48/49 \geq m_1 \geq 6/7$, the perfect Bayesian equilibrium would fit behavior better.