Applied Game Theory
APEC 8205

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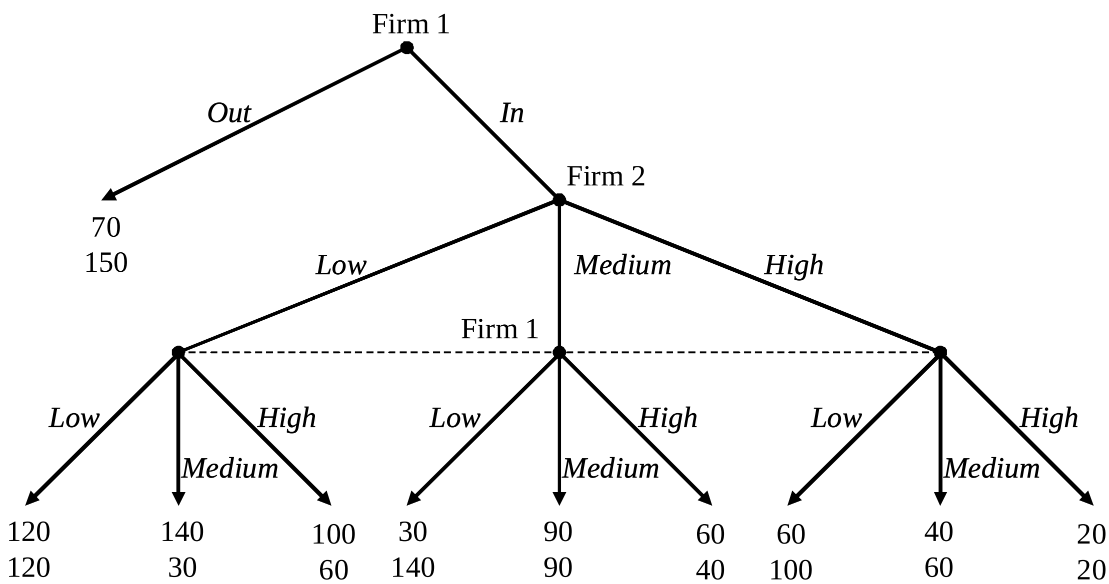
Midterm Exam Answers

1. Consider the game below. Firm 1 starts the game by choosing to enter the market (*In*) or to stay out (*Out*). If Firm 1 chooses to enter the market, Firm 1 and 2 must simultaneously choose whether to produce a ***Low***, ***Medium***, or ***High*** output. The top number is Firm 1's profit, while the bottom number is Firm 2's profit.
   a) List all of Firm 1's and Firm 2's strategies.
   b) Write this extensive form game in its normal form.
   c) Find all pure strategy Nash equilibria for the game.
   d) Which of these Nash equilibria is subgame perfect?
   e) Based on experimental evidence, do you believe that this subgame perfect equilibrium is a good prediction of behavior? Explain.



Firm 1's Profit
Firm 2's Profit

ANSWER:

a) Firm 1's strategies are {*Out*, *Low*|*In*, *Medium*|*In*, *High*|*In*}. Firm 2's strategies are { *Low*|*In*, *Medium*|*In*, *High*|*In* }.

b)

		Firm 2		
		Low|*In*	*Medium*|*In*	*High*|*In*
	Out	70, 150*	70, 150*	70*, 150*
Firm 1	*Low*|*In*	120, 120	30, 140*	60, 100
	Medium|*In*	140*, 30	90*, 90*	40, 60
	High|*In*	100, 60*	60, 40	20, 20

c) In the Table above, I have placed * by each player's best responses, which yields two pure strategy Nash equilibria: {*Out*, *High*|*In*} and {*Medium*|*In*, *Medium*|*In*}.

d) {*Medium*|*In*, *Medium*|*In*}: In the subgame starting at Firm 2's choice of output *High* is strictly dominated by *Medium*. Therefore, it is not part of a Nash equilibrium for this subgame and should be ruled out, which leaves {*Medium*|*In*, *Medium*|*In*}.

e) Firm 2 would give up anywhere from 30 to 40 by carrying through with a threat to play *High* if Firm 1 chooses *In*. The experimental evidence tells us that if carrying through with an incredible threat is not too costly, people may do it. Therefore, your answer here really depends on whether or not you think carrying through with this threat is too costly to Firm 2.

Several of you made a fairness argument that favored the prediction of the subgame perfect Nash equilibrium. This is also a reasonable argument given the experimental evidence that people often choose actions that may be against their interest, but are more equitable. But if this is what is happening, then {*Low*|*In*, *Low*|*In*} could be a more reasonable prediction than {*Medium*|*In*, *Medium*|*In*}.

2. Suppose two firms, denoted by 1 and 2, are competing for a lucrative government contract. The value of the contract, V, is the same for both firms. The government awards the contract to the firm that submits the highest quality proposal (if the proposals are of equal quality, neither firm gets the contract). The quality of a firm's proposal, q_i, is equal to the effort it puts into preparing the proposal, e_i, plus some random factor, e_i: $q_i = e_i + e_i$ where the density and distribution of e_i are $f(e_i)$ and $F(e_i)$ for $i = 1, 2$. Neither firm observes e_1 and e_2 before choosing effort, and e_1 and e_2 are independent. The cost of effort is $C(e_i) = e_i^2$.
   a) Find the symmetric Nash equilibrium effort assuming firm's seek to maximize the expected value of the contest.
   b) Assuming e_i for $i = 1, 2$ is normally distributed with mean 0 and variance s^2 (i.e. $f(e_i) = \frac{1}{\sqrt{2p}\,s} e^{-\frac{e_i^2}{2s^2}}$ for $\infty > e_i > -\infty$), what happens to equilibrium effort as the variance of the e_i increases? Explain the intuition of this result.

ANSWER:

a) The ith firm's optimization problem can be written as

$$\max_{e_i \geq 0} \Pr(q_i > q_j)V - C(e_i) = \Pr(e_i + e_i > e_j + e_j)V - C(e_i).$$

The first order condition for an interior optimum is

$$\frac{\partial \Pr(e_i + e_i > e_j + e_j)}{\partial e_i} V - C'(e_i) = 0.$$

Note that the $\Pr(e_i + e_i > e_j + e_j) = \Pr(e_i - e_j + e_i > e_j) = \int F(e_i - e_j + e_i) f(e_i) de_i$ such that $e^* = \frac{V \int f(e_i)^2 de_i}{2}$ when $e_1^* = e_2^* = e^*$.

b) For $f(e_i) = \frac{1}{\sqrt{2p}\,s} e^{-\frac{e_i^2}{2s^2}}$, $\int f(e_i)^2 de_i = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2p}\,s} e^{-\frac{e_i^2}{2s^2}} \right)^2 de_i = \int_{-\infty}^{\infty} \frac{1}{2ps^2} e^{-\frac{e_i^2}{s^2}} de_i =$

$\frac{1}{2ps^2} \sqrt{2p} \frac{s}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2p}\frac{s}{\sqrt{2}}} e^{-\frac{e_i^2}{2\frac{s^2}{2}}} de_i = \frac{1}{2\sqrt{ps^2}}$. Therefore, $e^* = \frac{V}{4\sqrt{ps^2}}$ such that $\frac{\partial e^*}{\partial s^2} = -\frac{V}{8s^2\sqrt{ps^2}} < 0$. As the quality of the proposal becomes less certain given a firm's effort, firm's invest less effort in preparing the proposal.

3. Medmax and Pliant Corp. are two firms that compete in the medical device industry. These firms engage in R&D that drives down their cost of production. They compete as Cournot duopolists by simultaneously choosing how much to produce. Let q_1 be the production of Medmax and let q_2 be the production of Pliant. The market inverse demand curve is given by $P(Q) = a - Q$, where P is the market price and Q is quantity supplied, with $Q = q_1 + q_2$. Assume that the production costs are given by $C(q_i) = (c_i - e_i)\, q_i$, for $c_i > e_i$ and zero otherwise, where e_i is R&D effort by firm i, $i = 1, 2$. Let the cost of R&D for firm i be given by $D(e_i) = d_i e_i^2$.
   a) Suppose firms simultaneously choose R&D and quantity all at the same time. Solve for Nash equilibrium.
   b) Suppose that firms simultaneously choose R&D in the first stage and then simultaneously choose quantity in the second stage after observing R&D expenditures. Solve for subgame perfect Nash equilibrium.
   c) How do the Nash equilibrium levels of R&D found in part (a) compare to the subgame perfect R&D levels found in part (b)? Explain why this result occurs.

   Note: For a Nash equilibrium in a Cournot game with constant marginal costs and linear inverse demand, the profits of a firm are equal to the square of quantity:

   $$p_i(q_i^*, q_j^*) = (q_i^*)^2$$

ANSWER:

a) Firms choose R&D and quantities at the same time so the problem for the firm is:

$$Max\ (a - q_i - q_j)q_i - (c_i - e_i)q_i - d_i e_i^2$$

which has two first order conditions:

$$a - 2q_i - q_j - (c_i - e_i) = 0$$
$$q_i - 2d_i e_i = 0$$

Use the second equation to solve for q_i and substitute this into the first equation to find:

$$a - c_i - (4d_i - 1)e_i + 2d_j e_j = 0$$

There are now two equations in two unknowns (e_1, e_2), solving for the Nash equilibrium R&D levels we find:

$$e_1^* = \frac{(4d_2 - 1)(a - c_1) - 2d_2(a - c_2)}{(4d_1 - 1)(4d_2 - 1) - 4d_1 d_2}$$
$$e_2^* = \frac{(4d_1 - 1)(a - c_2) - 2d_1(a - c_1)}{(4d_1 - 1)(4d_2 - 1) - 4d_1 d_2}$$

Using these expressions, we can find equilibrium quantities:

$$q_1* = 2d_1e_1* = 2d_1 \frac{(4d_2-1)(a-c_1)-2d_2(a-c_2)}{(4d_1-1)(4d_2-1)-4d_1d_2}$$

$$q_2* = 2d_2e_2* = 2d_2 \frac{(4d_1-1)(a-c_2)-2d_1(a-c_1)}{(4d_1-1)(4d_2-1)-4d_1d_2}$$

b) Finding subgame perfect equilibrium involves three steps:
- Solve for Nash equilibrium strategies in the second stage as a function of production cost (which depend on R&D)
- Substitute the Nash equilibrium strategies into the profit function to get payoffs only as a function of production costs
- Solve for Nash equilibrium strategy choices in R&D

Nash equilibrium in quantities:

Each firm faces the following problem:

$$Max\ (a-q_i-q_j)q_i-(c_i-e_i)q_i-d_ie_i^2$$

The first order condition for an optimal solution to this problem is:

$$a-2q_i-q_j-(c_i-e_i)=0$$

$$q_i(q_j)=\frac{a-(c_i-e_i)-q_j}{2}$$

Solving for the Nash equilibrium (i.e, where the best response functions cross), we find:

$$q_1* = \frac{a-2(c_1-e_1)+(c_2-e_2)}{3}$$

$$q_2* = \frac{a-2(c_2-e_2)+(c_1-e_1)}{3}$$

Substitute the Nash equilibrium strategies into the profit function

$$p_1(q_1*,q_2*)=(q_1*)^2-d_1e_1^2=\left(\frac{a-2(c_1-e_1)+(c_2-e_2)}{3}\right)^2-d_1e_1^2$$

$$p_2(q_1*,q_2*)=(q_2*)^2-d_2e_2^2=\left(\frac{a-2(c_2-e_2)+(c_1-e_1)}{3}\right)^2-d_2e_2^2$$

Solve for Nash equilibrium strategy choices in R&D

Taking the derivative of the profit function with respect to R&D, the first order condition for an optimal solution to this problem is:

$$\frac{4}{9}[a-2(c_i-e_i)+(c_j-e_j)]-2d_ie_i=0$$

$$e_i(e_j)=\frac{2(a-2c_i+c_j)-2e_j}{9d_i-4}$$

Solving for the Nash equilibrium (i.e, where the best response functions cross), we find:

$$e_1*=\frac{2(9d_2-4)(a-2c_1+c_2)-4(a-2c_2+c_1)}{(9d_1-4)(9d_2-4)-4}$$

$$e_2*=\frac{2(9d_1-4)(a-2c_2+c_1)-4(a-2c_1+c_2)}{(9d_1-4)(9d_2-4)-4}$$

One can then substitute these expressions into the expressions for q_1* and q_2* to find the subgame perfect equilibrium strategies for the players.

c) R&D levels in part (b) are higher than R&D levels in part (a). The easiest way to show this is to think about the logic of two-stage games. In part (b), if firm ***i*** increases e_i in the first stage, it will get firm ***j*** to decrease q_j in the second stage, which will increase firm i's profit. In part (a), since all moves are simultaneous, there can be no such reaction. Therefore, the strategic incentive to invest in R&D is not present and R&D investment will be lower. Since R&D levels are higher in part (b) than in part (a), costs will be lower and quantities will tend to be higher in part (b) than in part (a), and will necessarily be higher in the symmetric case.

The algebra of the comparison is messy in the general case, but is quite simple in a symmetric case where we let $c_1 = c_2 = c$ and $d_1 = d_2 = 1$. In this case, in part (b) we have:

$$e_1*=e_2*=e*=\frac{2(9-4)(a-2c+c)-4(a-2c+c)}{(9-4)(9-4)-4}=\frac{6(a-c)}{21}=\frac{2(a-c)}{7}$$

$$q_1*=q_2*=q*=\frac{a-c+e*}{3}=\frac{3(a-c)}{7}$$

In part (a) we have:

$$e_1*=e_2*=\frac{(4-1)(a-c)-2(a-c\ \ )}{(4-1)(4-1)-4}=\frac{(a-c)}{5}$$

$$q_1*=q_2*=q*=\frac{a-c+e*}{3}=\frac{2(a-c)}{5}$$

In this symmetric case, we see that equilibrium R&D and quantity produced are higher in part (b) than in part (a).