Math Review

Fundamental Algebraic Properties You Will Need to Know

Definition
The set of integers is {…,-4,-3,-2,-1,0,1,2,3,4,…}

Definition
Rational Number: Any number that can be written as an integer divided by an integer (e.g. ½ = 0.5, and -4/3 = -1.333…).

All integers are a rational numbers.

Definition
Irrational Number: Numbers that cannot be written as an integer divided by an integer (e.g. $\sqrt{2}$ = 1.4142…, and π = 3.1415…)

Irrational numbers have non-repeating and non-terminating decimals. Each irrational number falls between two rational numbers.

Definition
Real numbers: All rational and irrational numbers.

Properties of Real Numbers

Name	Description	Example
Closure	$a + b$	$1 + 2 = 3$
	ab are real numbers	$1 \times 2 = 2$ are real
Commutative	$a + b = b + a$	$1 + 2 = 2 + 1$
	$ab = ba$	$1 \times 2 = 2 \times 1$
Associative	$(a + b) + c = a + (b + c)$	$(1 + 2) + 3 = 1 + (2 + 3)$
	$(ab)c=a\,(bc)$	$(1 \times 2) \times 3 = 1 \times (2 \times 3)$
Distributive	$a(b + c)= ab + ac$	$3 \times (1 + 2) = 3 \times 1 + 3 \times 2$
	$(b + c)\,a= ba + ca$	$(1 + 2) \times 3= 1 \times 3 + 2 \times 3$
Identity	$a + 0=a$	$5 + 0 = 5$
	$a \times 1=a$	$5 \times 1 = 5$
Inverse	Additive: $a + (-a) = 0$	$5 + (-5) = 0$
	Multiplicative: $a\,(1/a)=1$	$5 \times (1/5) = 1$
Cancellation	If $a + b = a + c$, then $b = c$	
	If $ab = ac$ and $a \neq 0$, then $b = c$	
Zero Factor	$a \times 0=0 \times a=0$	
	$ab = 0$ if and only if $a = 0$, $b = 0$, or both	
Negation	$-1 \times (-a) = -\,(-a) = a$	
	$(-a)(-b) = ab$	
	$(-a)(b) = (a)(-b) = -(ab)$	

Note: a, b, and c are real numbers

During the course of the semester we will manipulate various algebraic expressions. Our knowledge of the above properties will be instrumental in accomplishing this task. Therefore, if you do not understand these properties, you should see me immediately.

Exponents/Powers of a Number

Let a and b be real numbers and m and n be integers: $a^n = aa\ldots a$ where a is multiplied by itself n times.

Properties of Exponents

$a^n a^m = a^{n+m}$	e.g. $4^2 \times 4^5 = 4^{2+5} = 4^7 = 16{,}384$.
$(a^n)^m = a^{nm}$	e.g. $(4^2)^5 = 4^{2\times 5} = 4^{10}$ = a really big number.
$a^n b^n = (ab)^{\mathrm{n}}$	e.g. $2^2 \times 3^2 = (2\times 3)^2 = 6^2 = 36$
$a^{-1} = 1/a$	e.g. $2^{-1} = \tfrac{1}{2} = 0.5$

Note that these properties also hold if n and m are real numbers, but the solution can be what is called a complex number. While I might choose n and m to be something other than integers, we will never deal with complex numbers in this class.

Functions

A function describes a definite relationship. For example, there is a definite relationship between the area of a circle, y, and its radius, x: $y = px^2$ or $y = f(x)$ where $f(x) = px^2$.

Often, the way a function is written denotes a causal relationship: y depends on x. The value y is then referred to as the dependent variable, while x is referred to as the independent variable. Just how y depends on x is described by $f(x)$.

But, this need not be the case. For example, we could have written the relationship between the area of a circle and the radius as $x = \sqrt{y/p} = f^{-1}(y)$ where $f^{-1}(y)$ is called the inverse of x [note that the inverse function has the property that $y = f(f^{-1}(y))$ for any y]. Now, if we know the area of a circle, we can figure out the radius using $f^{-1}(y)$.

For many of the functions we will deal with in this class, there will not always be a clear causal relationship or the causal relationship will depend on the particular set of circumstances being considered. For example, when we walk into a grocery store, the price of products is clearly marked and we choose how many to buy at that price. In this instance, price is the independent variable and the quantity we purchase is dependent on the price. However, if we decide to buy a new car, the price we pay depends on the type of car, its options, and our ability to negotiate effectively. In this case, the car, options, and our ability to negotiate are the independent variables and the price is the dependent variable.

The equation $y = f(x)$ is a concise description of a relationship, e.g. the relationship between the radius of a circle and the area. But there are other ways of describing this relationship. For example, we could use a table or a graph. In tabular form, we could write the description as:

Area of Circle	Radius of Circle
3.14	1.0
12.57	2.0
28.27	3.0
50.27	4.0
78.54	5.0

Figure 1 and 2 show a graphical description of the function and its inverse. In mathematics, the convention is to write the independent variable on the horizontal or x-axis and the dependent variable on the vertical or y-axis. But again, if there is no clear casual relationship, the choice of how we graph things can be arbitrary. In economics, we will frequently deal with the relationship between price and quantity. The convention used by most economists is to put the price on the vertical or y-axis and the quantity on the horizontal or x-axis. This can cause some confusion because as I mentioned before, price is not always interpreted as the dependent variable. Don't be confused, we can always write price as a function of quantity or quantity as a function of price, whichever is most convenient and consistent with our arguments.

Not all definite relationships we will look at are as simple as $y = f(x)$. For example, what if we want to know the number of words in a book. What does it depend on? If T is the total number of words, p is the number of pages, and n is the average number of words per page, then $T = pn = f(p, n)$. $T = f(p, n)$ says the number of pages in a book depends on two different things, the number of pages and the average number of words per page. Many of the relationships we will deal with this semester will be similar to this. The price we are willing to pay or the quantity we are willing to purchase will depend on lots of different things. For example, how much steak I buy for dinner depends on my income, the price of steak, and the price of mushrooms, since I like to smother my steak with mushrooms. Again, instead of writing the relationship in a compact functional form, we could also write it as a table or graph. For Example, we could write the relationship between the total number of words in a book, the total number of pages, and the average number of words per page in tabular form as:

		Number of Pages			
		50	100	150	200
Average	100	5,000	10,000	15,000	20,000
Number of	200	10,000	20,000	30,000	40,000
Words per	300	15,000	30,000	45,000	60,000
Page	400	20,000	40,000	60,000	80,000
	500	25,000	50,000	75,000	100,000

Figure 3 and 4 show a graphical description of the function. Figure 3 shows what is called a surface plot. It is three dimensional and particularly difficult to draw (at least for me). Figure 4 shows a contour plot, which is just as descriptive, but much easier to draw. Each contour corresponds to a different number of pages. The number of words per page is graphed on the horizontal or x-axis. The total number of words is graphed on the vertical or y-axis. Holding the total number of pages constant, we can focus our attention on the relationship between the average number of words per page and the total number of words by looking at a single contour. If we are interested in the relationship between the total number of words and the average number of words per page for a book with more or fewer pages, we need to look at a different contour.

Note that we could have plotted the number of pages on the horizontal or x-axis and had contours for each of the average number of words per page. In this way, we could focus our attention on the relationship between the total number of words and the number of pages, while holding the number of words per page constant.

When we look at functions of more than two variables (e.g. $C = f(z, p, n)$ where C is the total number of characters in a book, z is the average number of characters per word, p is the number of pages, and n is the average number of words per page) contour plots become even more useful because drawing surface plots in more than three dimensions is an impossible task.

Linear Functions

Linear functions all have a common form: $y = b + mx$. b is called the intercept because it is the value of y when $x = 0$. m is called the slope, which tells us how fast y changes as x changes: $m = \Delta y/\Delta x$, which means m is equal to the change in y (the rise) divided by the change in x (the run) (See Figure 5). When $m > 0.0$, y and x are directly related: increasing x increases y. When $m < 0.0$, y and x are inversely related: increasing x decreases y. When $m = 0.0$, y and x are unrelated: increasing x has no effect on y. Figure 6 shows some examples. Note that we can also write $x = -b/m) +(1/m)y$ where the intercept is now $–b/m$, which tells us the value of x when $y = 0$, and the slope is now $1/m$, which tells us how x changes as y increases. The two functions describe the exact same relationship. In the first, y is described in terms of x. In the second, x is described in terms of y. This is synonymous to the two different ways we wrote the relationship between the area of a circle and its radius. They are two different ways of saying the same thing. Which way is better depends on whether we already know x or whether we already know y.

Solving Two Linear Equations

Often, we will be interested in solving a system of two linear equations:
(1) $y = b_1 + m_1x$, and
(2) $y = b_2 + m_2x$.

What we will want to find is the value of x for which y is the same in both equations (1) and (2), which might be for example the $y = 10 + 2.5x$ and $y = 30 - 1.5x$ in Figure 6.

To do this algebraically, we can set the right-hand-side of equation (1) equal to the right-hand-side of equation (2):

$b_1 + m_1x^* = b_2 + m_2x^*$.

We can then use our algebraic properties to find what x equals:

$m_1x^* = b_2 + m_2x^* - b_1$	by the Cancellation Property
$m_1x^* - m_2x^* = b_2 - b_1$	by the Cancellation Property
$(m_1 - m_2)x^* = b_2 - b_1$	by the Distributive Property
$x^* = (b_2 - b_1)/(m_1 - m_2)$	by the Cancellation Property if $m_1 - m_2 \neq 0$.

If $(m_1 - m_2) = 0$, the lines will be parallel or equal and there will be no solution or any value of x will lead to the same value for y. In this class, we will assume $m_1 - m_2 \neq 0$ to avoid this unnecessary complication.

With our x value in hand, we can find the y value by substituting back into equation (1) or (2). Either way, we should get the same value for y. If we don't, we made a mistake.

$y^* = b_1 + m_1 \dfrac{b_2 - b_1}{m_1 - m_2}$	by the Substitution of x for $(b_2 - b_1)/(m_1 - m_2)$
$y^* = \dfrac{b_1(m_1 - m_2)}{m_1 - m_2} + m_1 \dfrac{b_2 - b_1}{m_1 - m_2}$	by the Inverse and Identity Properties
$y^* = \dfrac{1}{m_1 - m_2}(b_1m_1 - b_1m_2 + m_1b_2 - m_1b_1)$	by the repeated application of the Distributive Property
$y^* = \dfrac{m_1b_2 - b_1m_2}{m_1 - m_2}$	by the Cancellation and Distributive Properties

Now what does x^* and y^* tell us graphically? It tells us where the two functions intersect. For $y = 10+2.5x$ and $y = 30-1.5x$, $x^* = (30 - 10)/(2.5 - (-1.5)) = 20/4 = 5$ and $y^* = 10+2.5\times5 = 30-1.5\times5 = 22.5$ (See Figure 7).

Elementary Calculus

Microeconomics tries to explain how people make choices when they have unlimited wants, but limited means. Calculus and its application to the maximization of a function provide a useful tool for answering many of the questions economists like to ask.

Definition

The derivative of a function, $y = f(x)$, is equal to $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Graphically, the derivative of a function given x tells us the slope of the line that is tangent to the function at that value (See Figure 8).

So why is the slope of a line tangent to a function so important? It is important because it tells us where the function attains either a maximum or minimum value (See Figure 9 and 10). By solving the equation $f'(x^*) = 0$, we can find the maximum or minimum of a function. Substituting x^* back into $f(x)$, $y^* = f(x^*)$, tells us the maximized or minimized value of the function.

Being able to find the maximum or minimum of a function is useful, but it would be nice to know exactly which one we have found. This is accomplished by taking the derivative of the derivative of $f(x)$: $f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$, which we call the second derivative. If $f''(x) < 0$ for all x, we will know we have found the maximum for the function. If $f''(x) > 0$ for all x, we know we have found the minimum for the function.

Properties of Derivatives:
Let $f(x)$, $g(x)$, and $h(x)$ be three functions for which we can calculate derivatives.
(i) If $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$.
(ii) If $f(x) = g(x)\, h(x)$, $f'(x) = g'(x)h(x) + g(x)h'(x)$.
(iii) If $f(x) = h(g(x))$, $f'(x) = h'(g(x))\, g'(x)$.
(iv) If $f(x) = h(x)/g(x)$, $f'(x) = (h'(x)\, g(x) - h(x)g'(x))/g(x)^2$.

Most of the functions we will use in this class, can be solved with the properties above and the knowledge that for
(v) $f(x) = ax^n$, $f'(x) = nax^{n-1}$
where a and n are real numbers. For example, if $f(x) = 5x^3$, $f'(x) = 3(5x^{3-1}) = 15x^2$. Also note that for
(vi) $f(x) = a$, $f'(x) = 0$.

For example, lets take the function we have graphed in Figure 8, $f(x) = 10 + 15x - 2x^2$. We can write this function as the sum of three different and simpler functions: $f(x) = f_1(x) + f_2(x) + f_3(x)$ where $f_1(x) = 10$, $f_2(x) = 15x$, and $f_3(x) = -2x^2$. From (vi), we know $f_1'(x) = 0$. From (v), we know $f_2'(x) = 15$ and $f_3'(x) = -4x$. From (i), we know $f'(x) = 15 - 4x$. If we set $f'(x^*) = 15 - 4x^* = 0$ and solve for x^* we get 15/4, which looks right. We can assure ourselves that this is a maximum by calculating $f''(x)$: $f''(x) = f_4(x) + f_5(x)$ where $f_4(x) = 15$ and $f_5(x) = -4x$; from (vi), we know $f_4'(x) = 0$; from (v), we know $f_5'(x) = -4$; therefore, $f''(x) = -4 < 0$. We have a maximum, which is obvious from the graph, but maybe not so obvious from just looking at the function.

It should also be noted that the above principles generalize to functions with more than one variable. For example, if $y = f(x, z)$, we can take the derivative (also referred to as the partial derivative) of y with respect to x or z: $\frac{\partial f(x,z)}{\partial x} = f_x(x,z) = \lim_{h \to 0} \frac{f(x+h,z) - f(x,z)}{h}$ or $\frac{\partial f(x,z)}{\partial z} = f_z(x,z) = \lim_{h \to 0} \frac{f(x,z+h) - f(x,z)}{h}$. This partial derivative represents the slope of the line tangent to the contour of a function showing the relationship between y and x holding z constant or between y and z holding x constant.

For example, let $y = x^{0.4}z^{0.6} = f(x,z)$. Surface and contour plots are shown in Figures 11, 12, and 13.

From (v) holding z constant, the derivative of $f(x, z)$ with respect to x will be equal $\frac{\partial f(x,z)}{\partial x} = f_x(x,z) = 0.4x^{0.4-1}z^{0.6} = 0.4x^{-0.6}z^{0.6} = 0.4\left(\frac{z}{x}\right)^{0.6}$. If we evaluate this expression at x = 20 and z = 25, $f_x(20,25) = 0.457$, which is the slope of the tangent line that has been drawn in Figure 12. For any value of x, $f_x(x, z)$ tells you the slope of the contour at x holding z constant.

From (v) holding x constant, the derivative of $f(x, z)$ with respect to z will be equal $\frac{\partial f(x,z)}{\partial z} = f_z(x,z) = 0.6x^{0.4}z^{0.6-1} = 0.6x^{0.4}z^{-0.4} = 0.6\left(\frac{x}{z}\right)^{0.4}$. If we evaluate this expression at x = 50 and z = 15, $f_z(50, 15) = 0.549$, which is the slope of the tangent line that has been drawn in Figure 13. For any value of z, $f_z(x, z)$ tells you the slope of the contour at z holding x constant. While the analysis we do in this class can be accomplished without using partial derivatives, it requires much more work, so I encourage all of you to become familiar with how to use partial derivatives.

The derivative allows us to identify the maximum or minimum of a function. Partial derivatives accomplish the same thing when there is more than one dependent variable. But instead of having one equation and one unknown, there will be several equations with several unknowns. These equations can be solved just like linear equations. To know for sure whether we found a maximum or a minimum, we need to look at the second derivatives. With two independent variables, there will be four second derivatives. The appropriate conditions for these second derivatives can be found in a good calculus book so we will not dwell on them here or during the rest of the class.

Figure 1: Relationship between the area and radius of a circle.

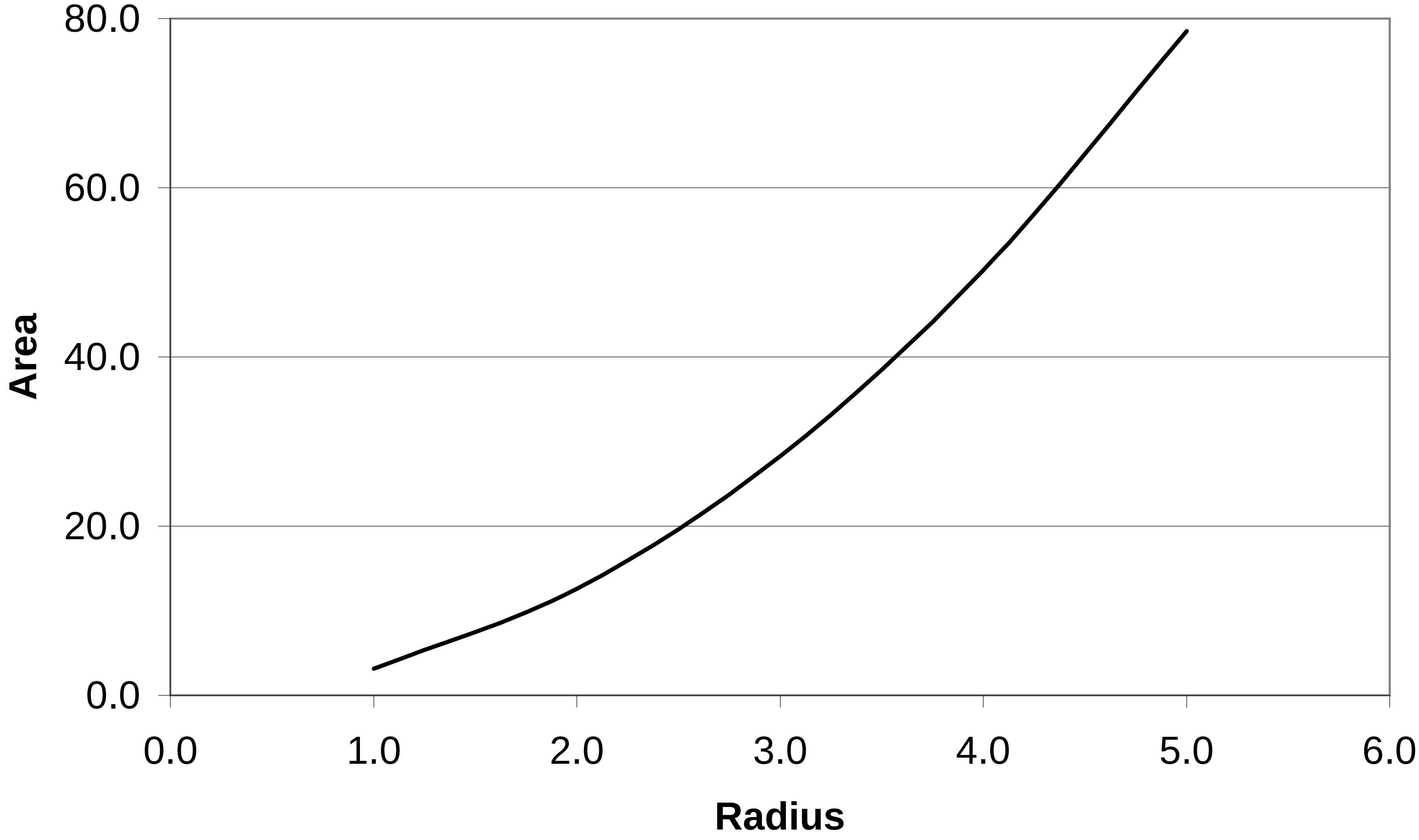


Figure 2: Relationship between the area and radius of a circle.

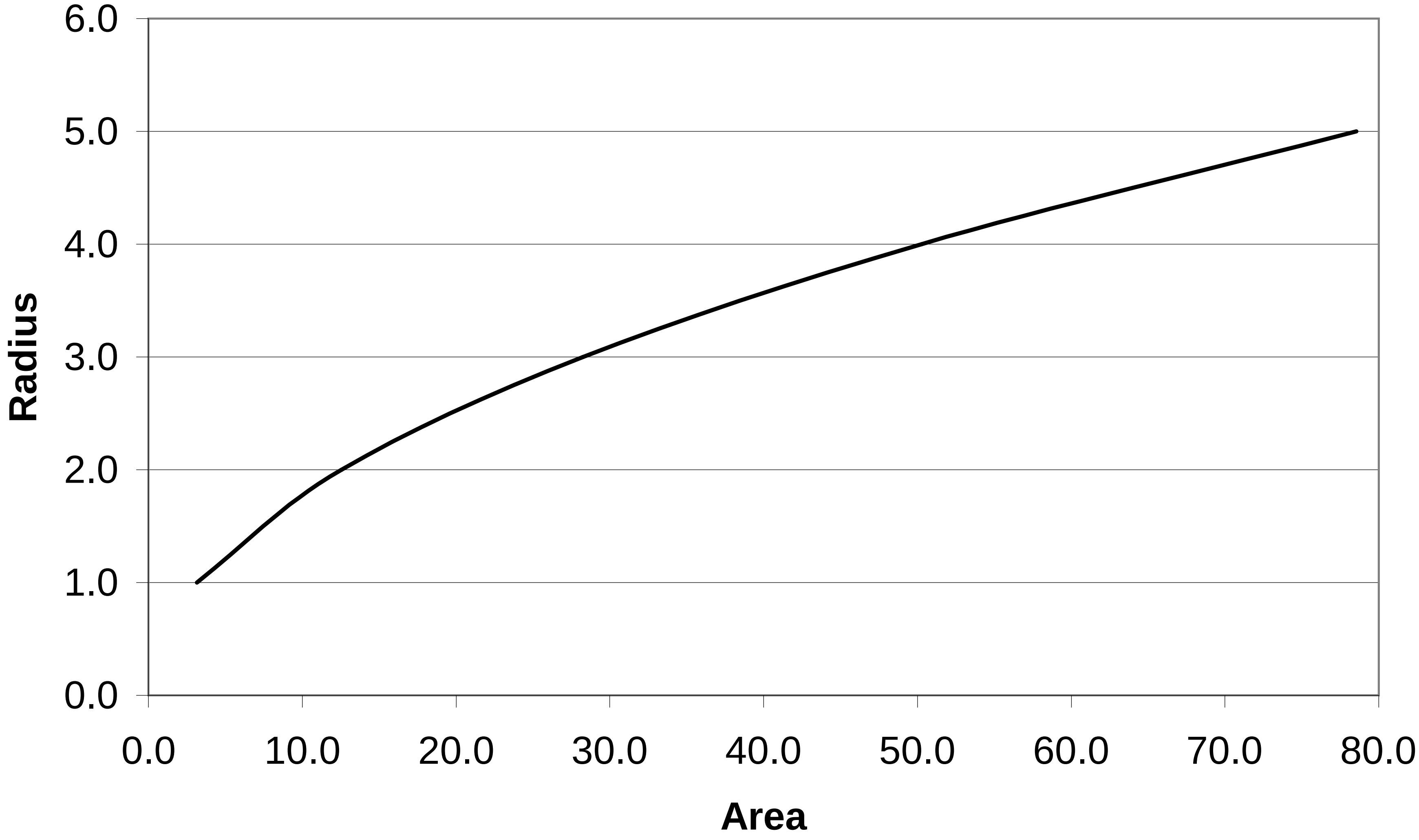
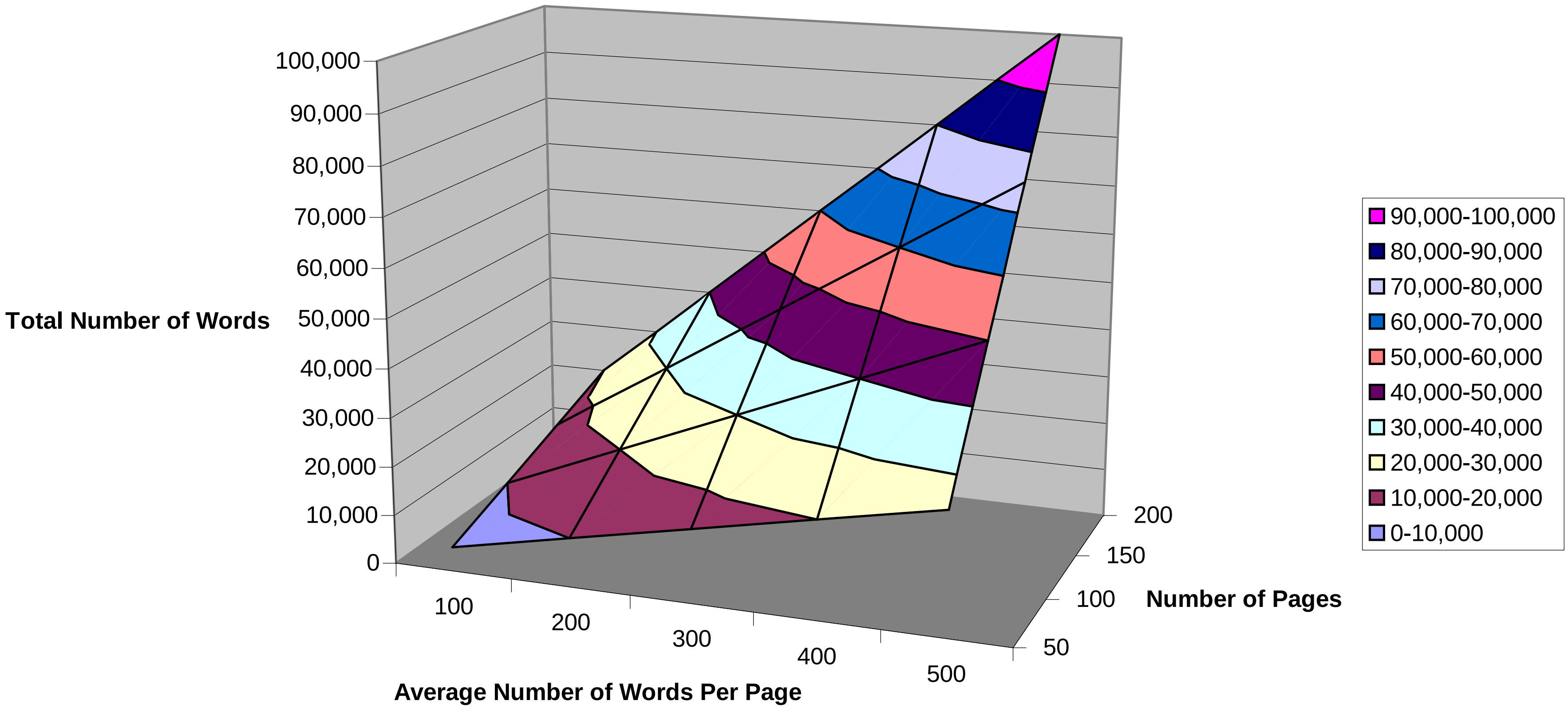


Figure 3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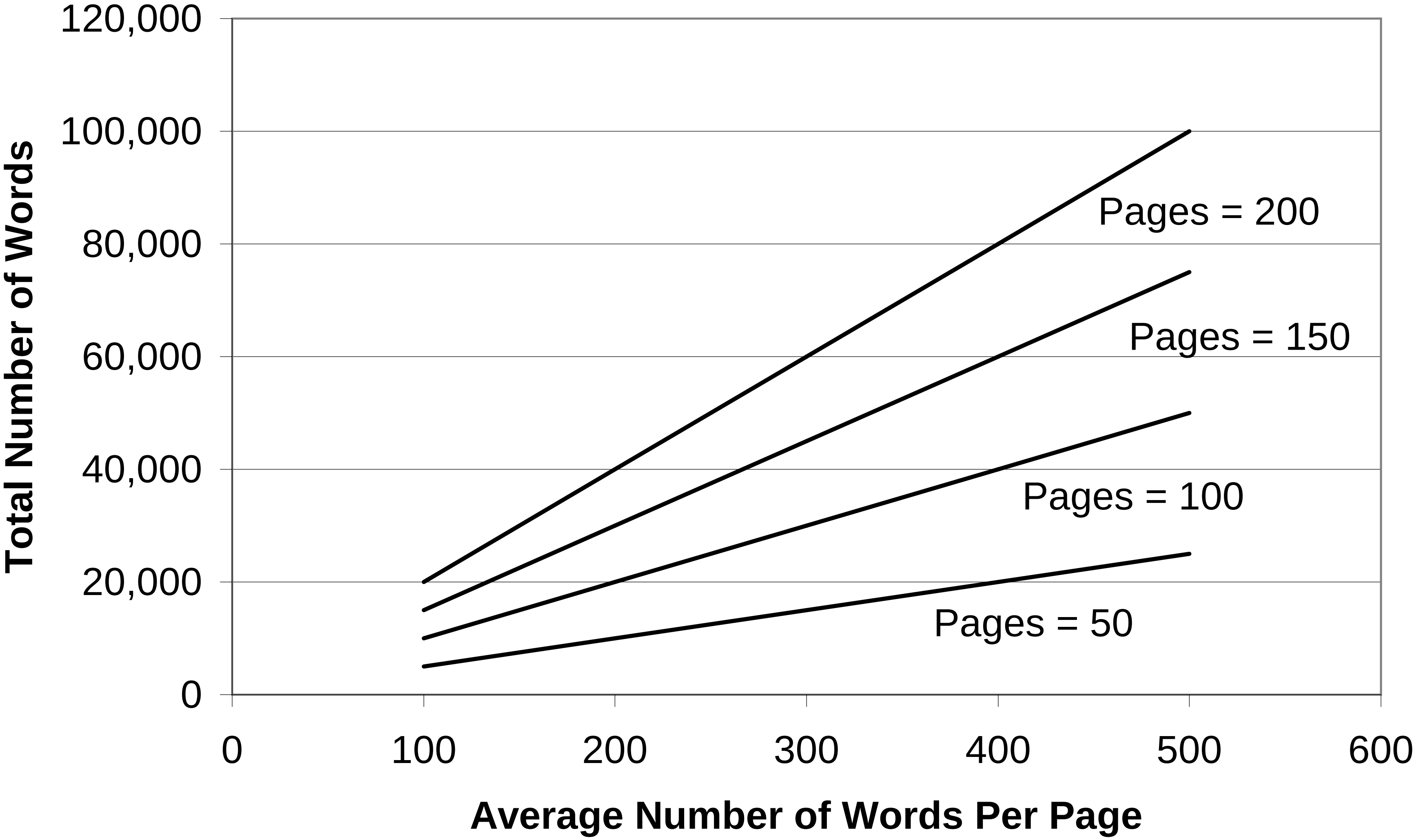
90,000
80,000
70,000
60,000
50,000
40,000
30,000
20,000
10,000
0
100
200
300
400
500
Average Number of Words Per Page
50
100
150
200
Number of Pages
90,000-100,000
80,000-90,000
70,000-80,000
60,000-70,000
50,000-60,000
40,000-50,000
30,000-40,000
20,000-30,000
10,000-20,000
0-10,000

Figure 4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100,000
80,000
60,000
40,000
20,000
0
Total Number of Words
Pages = 200
Pages = 150
Pages = 100
Pages = 50
0
100
200
300
400
500
600
Average Number of Words Per Page

Figure 5

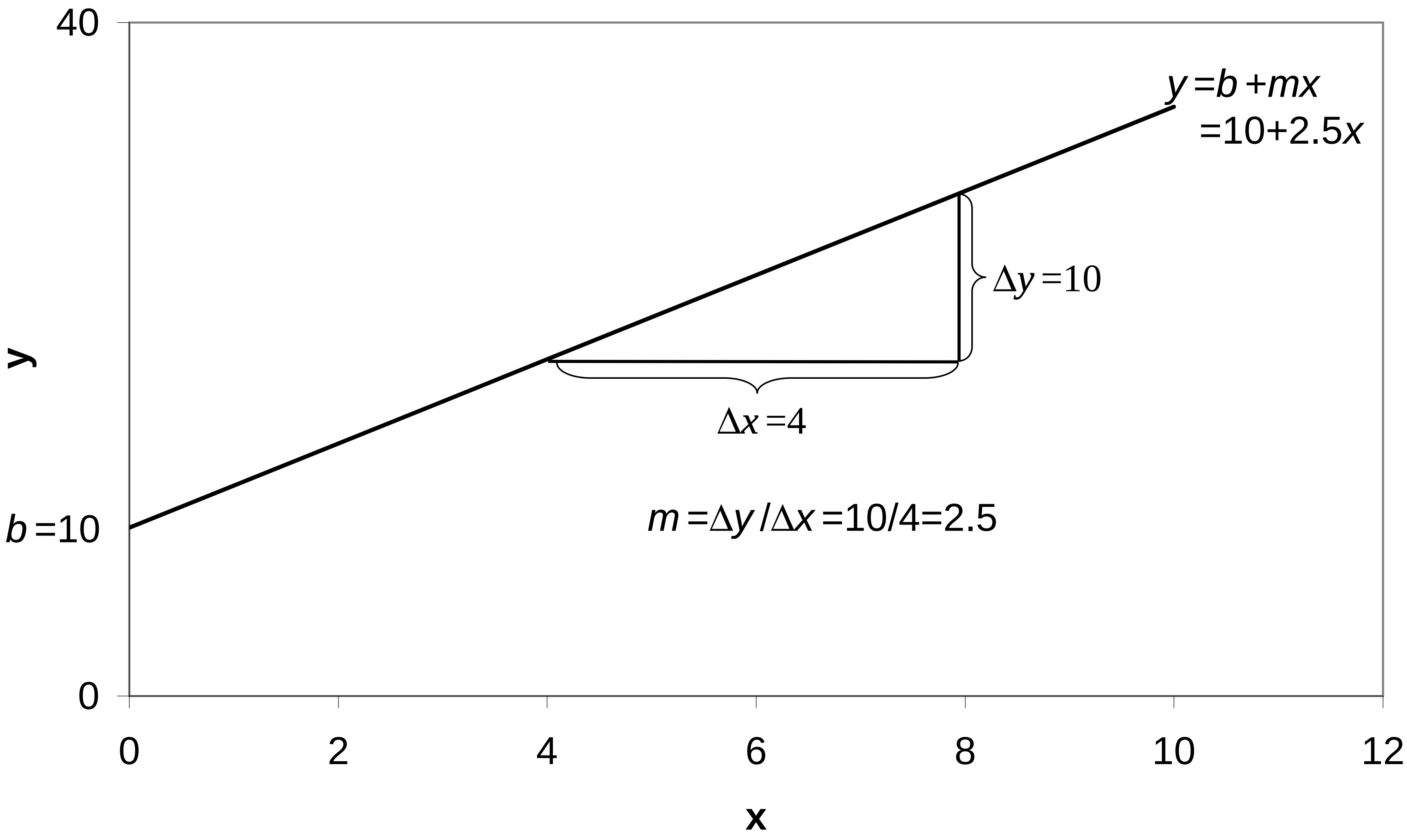


Figure 6

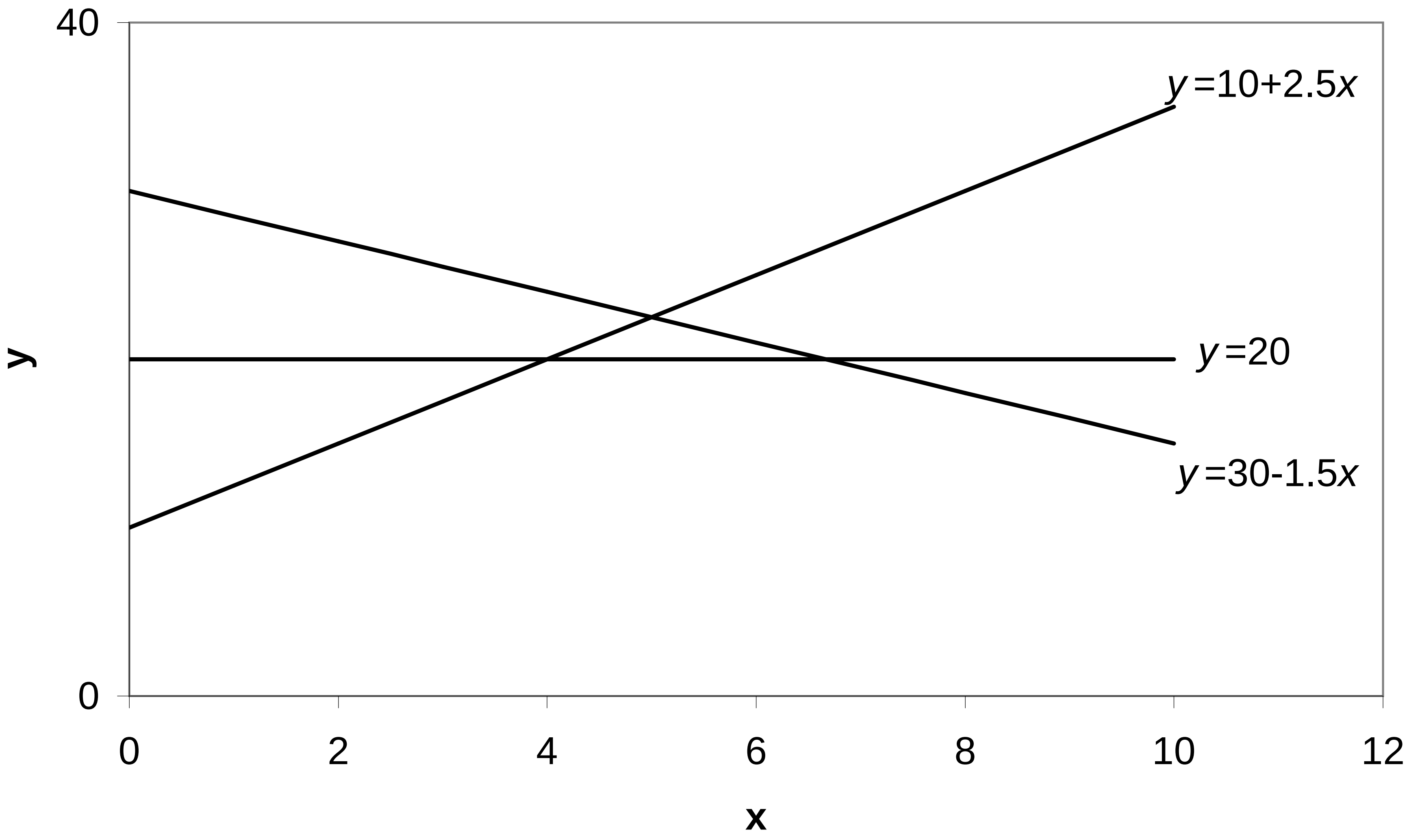


Figure 7

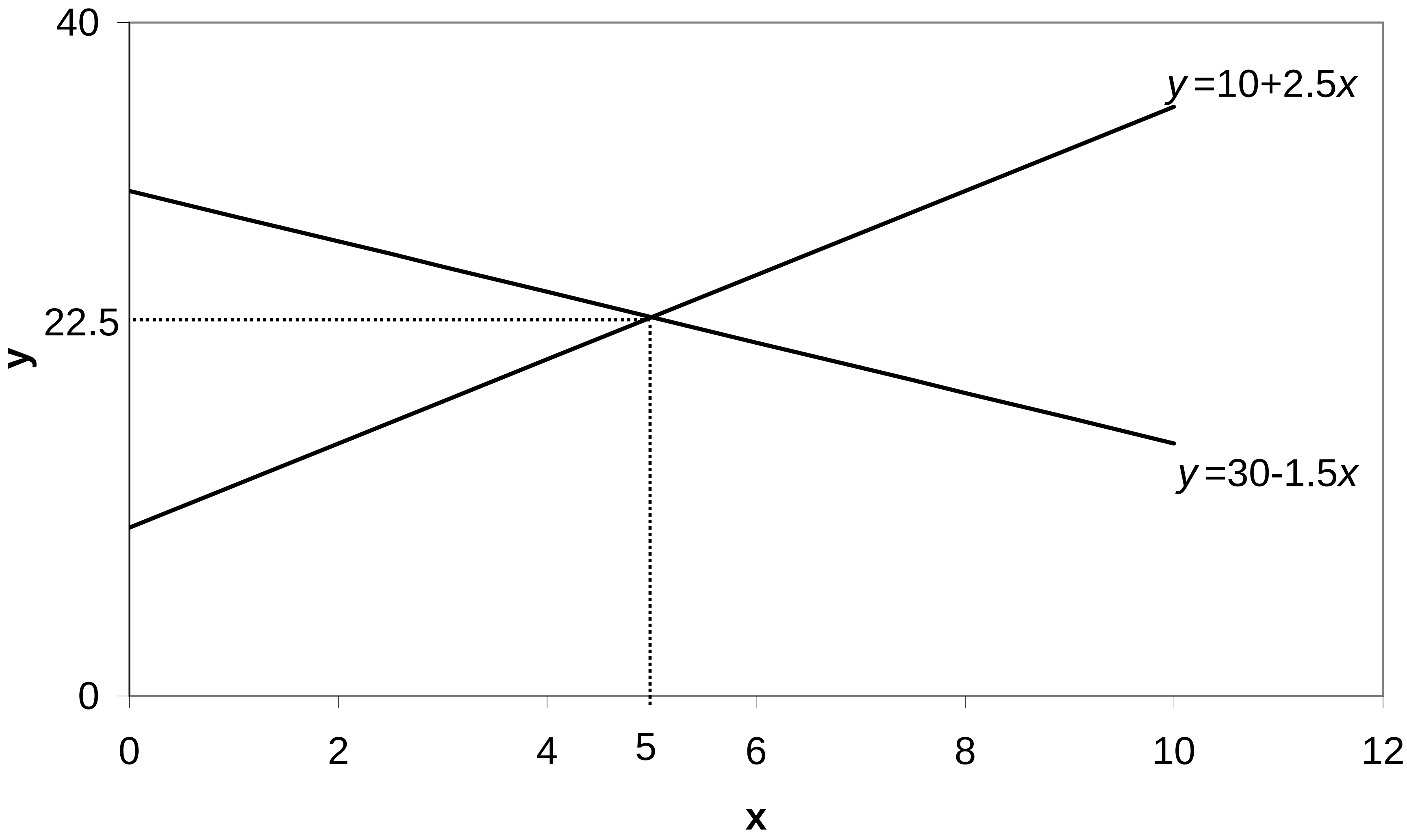
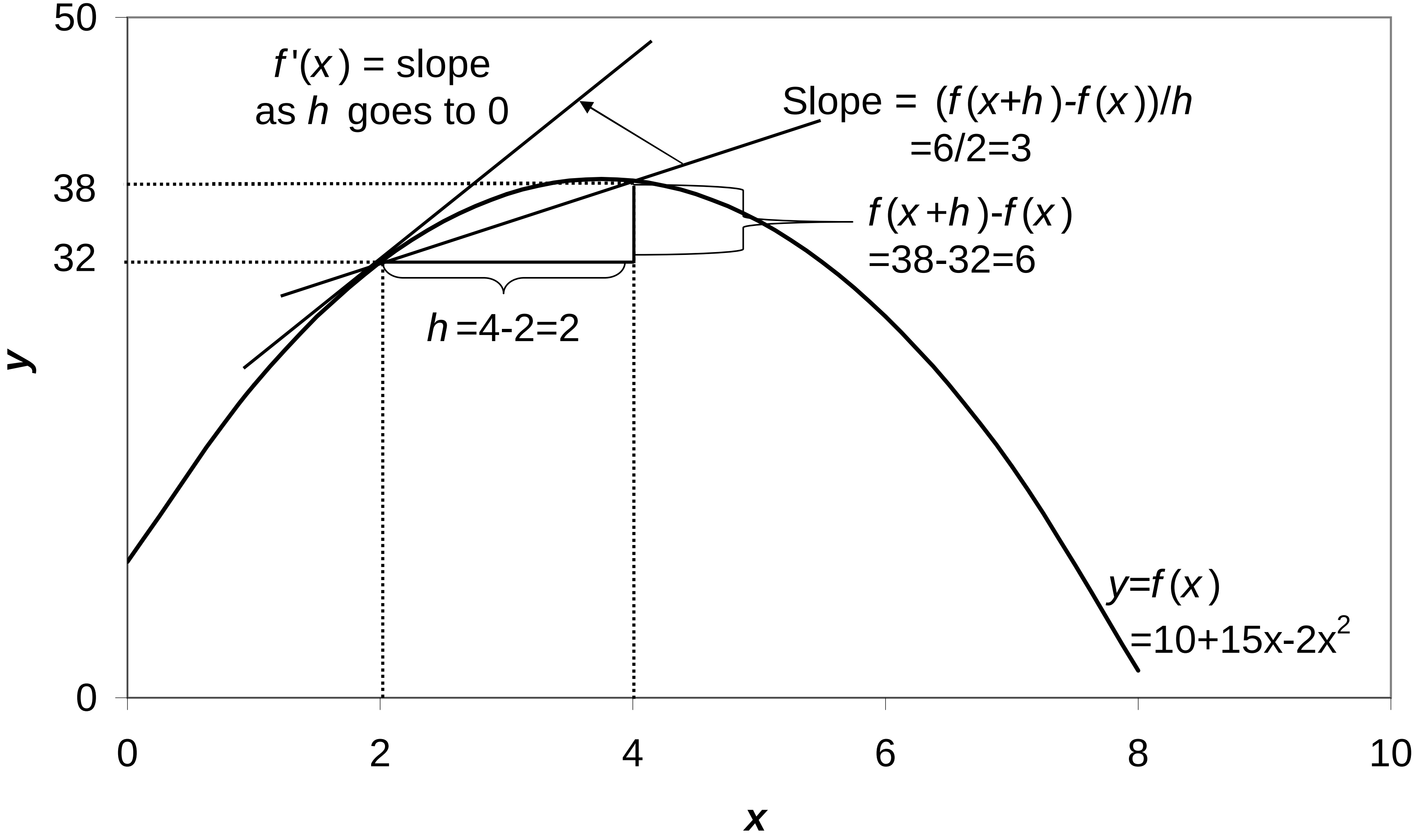


Figure 8



50
38
32
0
y
0
2
4
6
8
10
x
$f'(x)$ = slope
as h goes to 0
Slope = $(f(x+h)-f(x))/h$
=6/2=3
$f(x+h)-f(x)$
=38-32=6
h=4-2=2
$y=f(x)$
$=10+15x-2x^2$

Figure 9

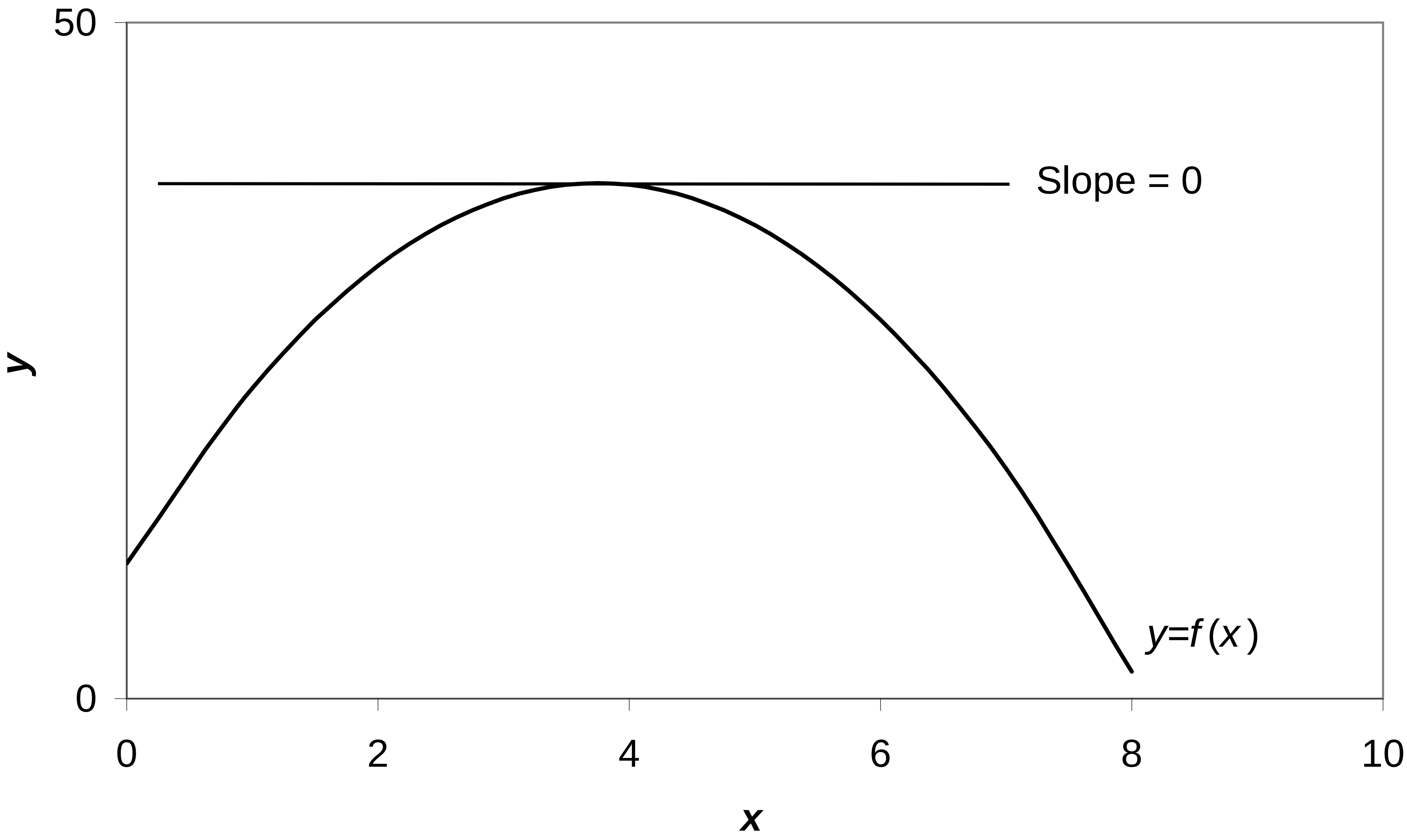


Figure 10

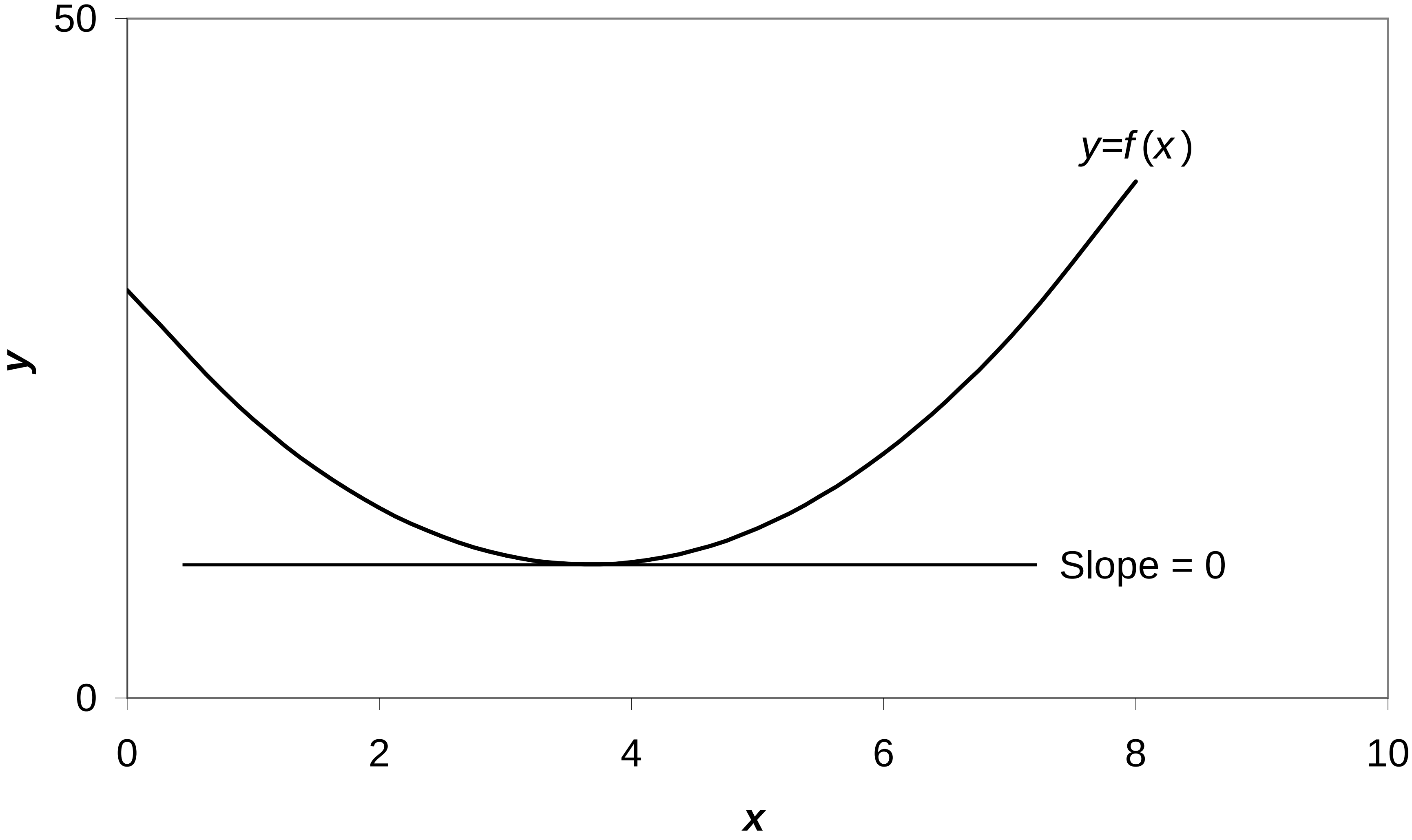
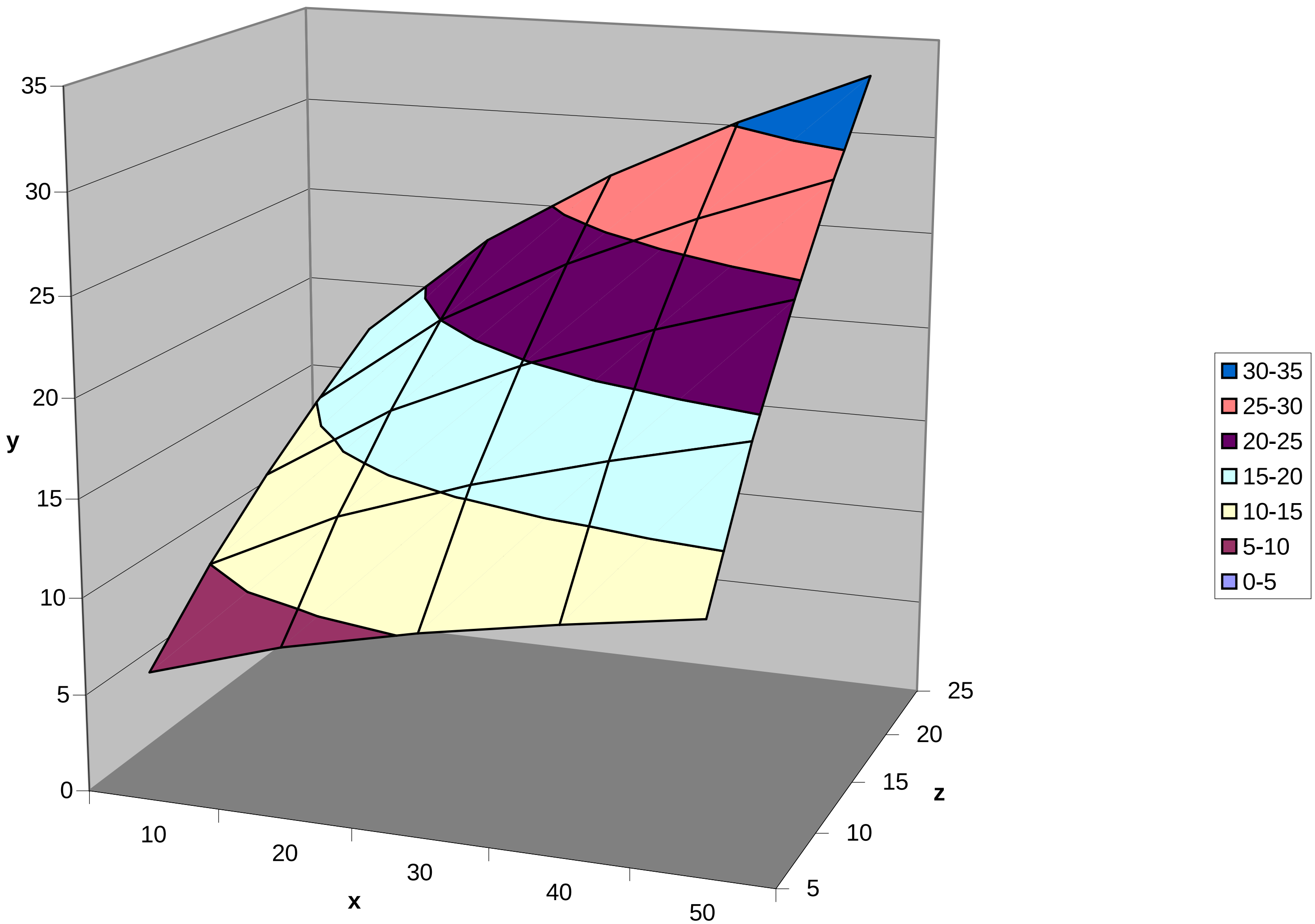


Figure 11


y
35
30
25
20
15
10
5
0
x
10
20
30
40
50
z
5
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15
20
25
30-35
25-30
20-25
15-20
10-15
5-10
0-5

Figure 12: Contour plots holding Z constant.

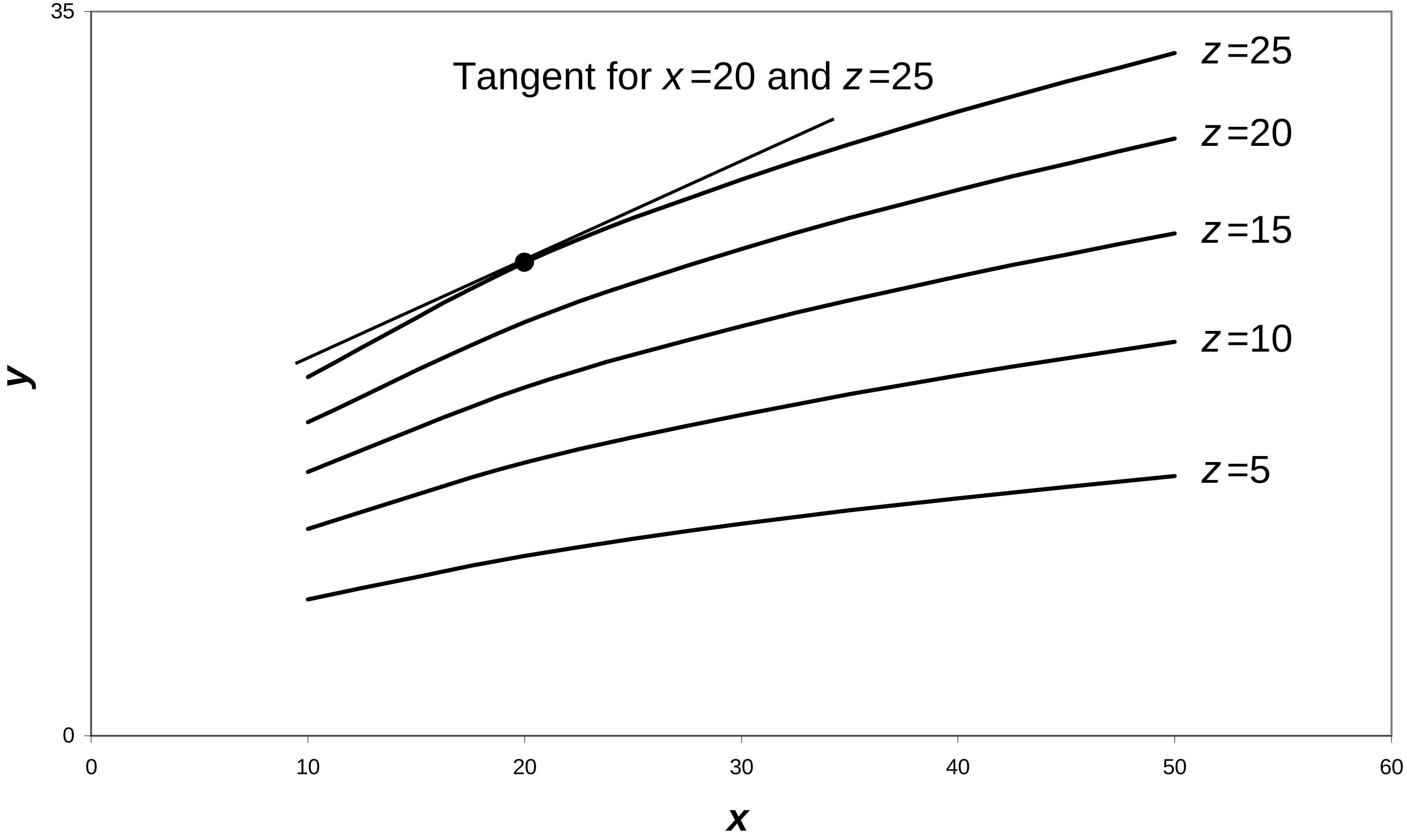

Figure 13: Contour plots holding x constant.